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OF MATHEMATICAL SCIENCE

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AN ANNOUNCEMENT

An announcement of the new forms in which the Bulletin is to appear has been made on pages 1–2 of the January issue. In each of the odd-numbered issues there will appear reports of meetings, reviews, notes, and abstracts of papers.

One of the announced purposes of the new forms is to secure earlier publication of the abstracts than has been possible under the previous arrangements. Eventually it will be possible to publish many abstracts in advance of the meetings at which the corresponding papers are read, and it is hoped that a beginning can be made immediately, in so far as the authors are prepared to send their abstracts to the Secretaries within a time-limit which will permit of publication in advance. The Secretaries will issue from time to time notices of the exact dates before which abstracts must be presented if the authors desire that they be published in advance of the meeting for which they are intended.

The March issue will contain all of the abstracts of the papers read at the February meeting, and as many as possible of the abstracts which are in the hands of the Secretaries on or before March 7 for papers to be read at the three April meetings to be held in Stanford University, Chicago, and New York, respectively, on April 5, April 18–19, and April 18–19. It is hoped that that issue can be in the hands of members not later than April 15.

The present issue is typical of the even-numbered issues. They will contain scientific papers, including longer addresses, and the list of new publications. The paging is continuous with the January issue. In all, twelve issues will appear, alternately of the types mentioned above. The appearance of the bound volume will not be materially altered, and the size of the volume will not be affected.

The editors wish to repeat also the invitation to members and to other readers, to offer comments and suggestions regarding the present forms.

THE EDITORS

NON-EUCLIDEAN GEOMETRY, A RETROSPECT*

BY JAMES PIERPONT

1. *Introduction.* Non-euclidean geometry had its origin in the unwearied attempts of mathematicians during 2000 years to free the *Elements* from its one notorious blemish, namely, the postulate or axiom relating to parallel lines.

Wallis (1693), Saccheri (1733), Lambert (1786), Legendre (1794), Schweikert (1807), Wachter (1817), and Taurinus (1826) are noteworthy forerunners, but the first systematic and rigorous development of the subject to be published was a series of five papers entitled *On the principles of geometry*, which appeared during 1829–30 in the Kasan Messenger. Their author was an unknown Russian mathematician, Lobachevsky. The present year 1929 may be regarded as rounding out the first century of this new science. What has it accomplished in this time? I propose to answer this question, paying attention not so much to concrete results obtained, as to the basal ideas which have made its phenomenal progress possible.

The early history of our subject is too well known to require more than a few words. We must mention, however, that Gauss was already in full possession of Lobachevsky's results, although he permitted nothing to reach the ears of the public. The Hungarian Bolyai had also broken through the barriers of euclidean geometry independently. His treatise, through no fault of his own, did not appear till 1832.

The method used by these geometers was the synthetic method of Euclid, with a slight mixture of trigonometry, analytic geometry and the calculus. It was adequate to establish the existence of a non-euclidean geometry, but new methods and ideas were necessary for further progress. These came with a bound.

Riemann's Habilitationsschrift *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, read before the Philosophical Faculty of the University of Goettingen in 1854, but first

* An address delivered by invitation of the Program Committee at the meeting of the Society in Berkeley, June 30, 1929.

published in 1868, after Riemann's death; Cayley's *Sixth memoir upon quantics* (1859); Klein's *Ueber die sogenannte Nicht-Euklidische Geometrie* (1871): these were the epoch making papers of this second period, which we now consider.

2. *The Projective Methods of Cayley and Klein.* Cayley takes four numbers x_1, x_2, x_3, x_4, and regards their ratios $x_1:x_2:x_3:x_4$ as defining a point. In the notes to volume 2 which Cayley prepared for his *Collected Papers*, he says on page 605:

"As to my memoir, the point of view was that I regarded 'coordinates' not as distances or ratios of distances, but as an assumed fundamental notion not requiring or admitting of explanation."

This abstract method of procedure seems to have been entirely misunderstood by Klein and as a result he has developed non-euclidean geometry from another point of view, as we shall see.

Continuing with Cayley, we define a straight abstractly by the points

$$x_i = la_i + mb_i, \qquad (i = 1,2,3,4).$$

Here l, m are parameters and a_i, b_i the coordinates of two points a, b. A plane is defined by the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0.$$

To define distance and angle, Cayley introduces the quadratic form

$$F(x,x) = (x,x) = \sum a_{ij}x_ix_j, \quad a_{ij} = a_{ji}, \quad (i,j = 1,2,3,4),$$

and the bilinear form

$$F(x,y) = (x,y) = \sum a_{ij}x_iy_j.$$

In euclidean geometry, $F(x, x) = 0$ defines a quadric surface which Cayley calls the *absolute*.

Associated with the form F is the form

$$G(u,u) = \sum a^{ij}u_iu_j,$$

in which a^{ij} = minor a_{ij}/a, where a denotes the determinant $|a_{ij}|$. Then $G=0$ is the equation of the absolute in plane coordinates. Cayley now defines the distance between two points x, y by means of the formula

$$\cos\delta = \frac{F(xy)}{(F(xx)F(yy))^{1/2}}, \tag{1}$$

and the angle between two planes u, v by the formula

$$\cos\phi = \frac{G(u,v)}{(G(uu)G(vv))^{1/2}}. \tag{2}$$

With these definitions and those which naturally flow from them, it is possible to construct a complete and simple development of non-euclidean geometry.*

We turn now to Klein. He tells us in his notes to his papers on non-euclidean geometry (Mathematische Abhandlungen, vol. 1, pp. 50–52, 241–243) that he was introduced to this subject while attending Kummer's and Weierstrass' seminar in Berlin (1869–70) by O. Stolz, who was thoroughly familiar with the work of Lobachevsky, Bolyai, and von Staudt.

Klein soon saw the relation between the geometry of Lobachevsky and Bolyai and Cayley's memoir of 1859; and in 1871 he published his first paper on this subject, as already noted. His Goettingen lectures on *Nicht-Euklidische Geometrie* were published in autograph form in 1892. These have enjoyed the most widespread popularity, and probably the majority of mathematicians of the younger generation learned their non-euclidean geometry from these lectures.

Klein apparently was influenced by a youthful paper of Laguerre (1853) which contains implicitly this result: The angle ϕ made by two straights a, b meeting at the point x is expressed by $(i/2)\log(ab, u'u'')$, where u', u'' are the straights joining x with the cyclic points, and $(ab, u'u'')$ is the cross ratio of the four straights.

The equation of u', u'' is $u_1^2+u_2^2=0$, a degenerate conic. Following Cayley, Klein replaces this conic by the general conic $F=\sum a_{ij}x_ix_j=0$ in point coordinates, or $G=\sum a^{ij}u_iu_j=0$ in line coordinates; and he defines the distance δ between two points x, y in the plane by the formula

$$\delta = \frac{c}{i}\log(xy, x'x''). \tag{3}$$

* See the forthcoming paper by the author, *Cayley's definition of non-euclidean geometry.*

Here x', x'' are the points of intersection of the straight lx_i+my_i with the conic $F=0$, $(xy, x'x'')$ is the cross ratio of the points in the parenthesis, and c is a constant chosen so that δ may be real. Similarly the angle ϕ between two straights u, v, meeting at point x, is defined by the equation

$$\phi = \frac{c'}{i} \log (uv, u'u''). \tag{4}$$

Here u', u'' are the tangents drawn from x to the conic F or G, $(uv, u'u'')$ is the cross ratio of the straights in the parenthesis, and c' is a constant chosen so that ϕ may be real.

If we take $F=0$ to be $x_1^2+x_2^2+x_3^2=0$, we get the geometry of Riemann, which we shall discuss later; if we take $F=0$ to be $x_1^2+x_2^2-x_3^2=0$, we get the geometry of Lobachevsky and Bolyai. The extension of the foregoing to space is obvious. The expressions (3), (4) may be immediately transformed into those of (1), (2).

It appears at first sight as if little had been gained. Klein's real contribution is the following. The cross ratio $(xy, x'x'')$ in (3) depends on the coordinates of the four points x, y, x', x'', and also on the lengths of their segments. The expression (3) therefore defines a non-euclidean geometry by means of euclidean geometry. This would be a serious defect if it could not be obviated. This Klein has done employing the ideas of von Staudt, freed from the parallel axiom. If one wishes to carry this work through rigorously, it requires a very considerable effort which most mathematicians, I fancy, would be glad to avoid, as in fact I believe they do. It seems strange therefore that the simpler method of Cayley has been entirely overlooked. It should be noted that the superiority of the methods of Cayley and Klein over the synthetic methods of the founders of non-euclidean geometry is exactly analogous to the superiority of projective methods over the synthetic methods of Euclid in euclidean geometry. Another enormous advantage is due to the fact that many of the methods and results of euclidean projective geometry may be taken over in toto, or with obvious modifications, without putting pen to paper.

3. *The Differential Method. First Period. Riemannian Geometry.* This period begins with the publication (1868) of

Riemann's Habilitationsschrift mentioned above. Riemann's paper may be regarded as a vast generalization of Gauss' *Disquisitiones generales circa superficies curvas* (1827). It is interesting to note that Gauss was present when Riemann read his paper. Riemann's memoir, when finally published, was promptly taken up. Beltrami (1868), Christoffel and Lipschitz (1869), Schlaefli (1871), Beez (1874), Voss (1880), Ricci (1884), and Killing (1885) were among the first to enter this field. In 1901 Ricci and Levi-Civita published in volume 54 of the Mathematische Annalen a resumé of their new tensor analysis.

Riemann's point of departure is analogous to that of Cayley. Any n numbers $x_1, x_2, \cdots, x_n$ he regards as the coordinates of a point. When the x's vary in a continuous manner, the corresponding points range over a part or all of an n-way space. Riemann offers no geometric picture of his configurations; they are abstractions determined by arithmetic relations between the coordinates, *which latter are undefined.*

Riemann regards this highly abstract procedure as quite necessary, for at the close of his paper (Werke, p. 268, edition of 1876) he says: "Solche Untersuchungen, welche, wie hier ausgeführt, von allgemeinen Begriffen ausgehen, können nur dazu dienen, dass diese Arbeit nicht durch die Beschränktheit der Begriffe gehindert und der Fortschritt im Erkennen des Zusammenhangs der Dinge nicht durch überlieferte Vorurtheile gehemmt wird."

We cannot go into details, as that would take too much space,* but a few words may be spent on a fundamental question. Riemann regards the metric properties of his space to be defined by the distance between two nearby points $x, x+dx$. He takes this to be defined by the formula

$$(5) \qquad ds^2 = \sum a_{ij} dx_i dx_j, \qquad a_{ij} = a_{ji}, \qquad (i,j = 1,2,\cdots,n),$$

where the a's are functions of $x_1, \cdots, x_n$ whose determinant a is different from zero. For euclidean space of three dimensions, we would have

$$(6) \qquad ds^2 = dx_1^2 + dx_2^2 + dx_3^2 .$$

* The reader may if he chooses consult the author's paper, *Some modern views of space*, this Bulletin, vol. 32 (1926), pp. 225–258.

A curve in Riemann space is defined by n relations of the type $x_i = x_i(t)$; a straight is a curve along which $\delta \int ds = 0$, where ds is defined by (5).

When the variation is executed we are led to n differential equations defining the straight, namely,

$$(7) \qquad \frac{d^2x_i}{ds^2} + \sum_{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} \frac{dx_\lambda}{ds} \frac{dx_\mu}{ds} = 0, \quad (i,\lambda,\mu = 1,2, \cdots, n).$$

The symbols under the summation sign were introduced by Christoffel, who was one of the earliest to elaborate the ideas of Riemann. He sets

$$(8) \qquad \left[\begin{matrix} \alpha\beta \\ i \end{matrix} \right] = \frac{1}{2} \left(\frac{\partial a_{\alpha i}}{\partial x_\beta} + \frac{\partial a_{\beta i}}{\partial x_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_i} \right).$$

If we denote by $a^{\lambda\mu}$ the minor of $a_{\lambda\mu}$ in a, divided by a, we have

$$(9) \qquad \left\{ \begin{matrix} \alpha\beta \\ \lambda \end{matrix} \right\} = \sum_i a^{i\lambda} \left[\begin{matrix} \alpha\beta \\ i \end{matrix} \right].$$

In the following we shall set

$$(10) \qquad \left[\begin{matrix} \alpha\beta \\ i \end{matrix} \right] = C_{\alpha\beta,i}, \qquad \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} = C^i_{\alpha\beta}.$$

The quantities $\xi_\lambda = dx_\lambda/ds$ we call the direction parameters of the straight. Let ξ_λ, η_λ be the parameters of two straights meeting at a point. The straights $l\xi_\lambda + m\eta_\lambda$, where l and m are parameters, define a plane. Other geometrical terms are introduced in a similar manner, as generalizations of the corresponding notions in euclidean geometry.

The question at once arises: when do two quadratic differential forms of the type (5) define the same geometry? It is well known that the geometry on a cylinder or a cone, or more generally on a developable surface, is, for not too large regions, identical with that of the plane, that is, it is euclidean. More generally, we may ask: when is the geometry on two given surfaces the same (for not too extended regions)? This question was partly answered by Gauss by means of the important notion of total curvature or Gaussian curvature k, at a point on a surface. If the geometries of two surfaces have constant curva-

ture, this condition is sufficient. Gauss established the important fact that k is an invariant of the quadratic form

$$c_{11}du_1^2 + 2c_{12}du_1du_2 + c_{22}du_2^2, \tag{11}$$

defining the metric of the surface. The determinant of this form we will call c. We can write down the value of k in a neat form by introducing what is now known as the Riemannian curvature tensor. For the general quadratic form (5), this tensor has n components (not all distinct) defined by the equations

$$R_{\mu\lambda,jk} = \frac{\partial C_{\lambda j,\mu}}{\partial x_k} - \frac{\partial C_{\lambda k,\mu}}{\partial x_j} + \sum_{i\alpha}(C_{\lambda k,\alpha}C_{\mu j,i} - C_{\lambda j,\alpha}C_{\mu k,i})a^{i\alpha}, \tag{12}$$

where the C's are the Christoffel symbols (10) relative to the form (5), and $i, j, \alpha, \lambda, \mu = 1, 2, \cdots, n$. If we denote by $R_{\mu\lambda,jk}(c)$ the components of the Riemannian tensor relative to the binary form (11), the Gaussian curvature of a surface in euclidean geometry is

$$k = R_{21,12}(c)/c. \tag{13}$$

Guided by the ideas of Gauss, as developed in the *Disquisitiones generales*, Riemann proceeds as follows. Any two geodesics ξ, η in his general n-space (5) meeting at a point P define a plane $\bar{\omega} = l\xi + m\eta$. This being a two-way manifold, the coordinates $x_1, x_2, \cdots, x_n$ of any point on it are functions of two parameters, say u_1, u_2. Hence

$$dx_i = \frac{\partial x_i}{\partial u_1}du_1 + \frac{\partial x_i}{\partial u_2}du_2, \quad (i = 1, 2, \cdots, n).$$

These set in (5) show that the metric of the plane $\bar{\omega}$ is defined by a form of the same type as (11), say by

$$ds^2 = b_{11}du_1^2 + 2b_{12}du_1du_2 + b_{22}du_2^2, \tag{14}$$

whose determinant call b. The coefficients of this form depend on u_1, u_2, and the $2n$ direction parameters $\xi_1, \cdots, \xi_n$, $\eta_1, \cdots, \eta_n$.

Riemann defines the curvature of $\bar{\omega}$ to be

$$k = R(b)_{21,12}/b, \tag{15}$$

where the $R(b)$ here refers to the form (14). We see that this is

entirely analogous to Gauss' definition (13). Here, however, the surface $\tilde{\omega}$ lies in an n-way space whose metric is given by (5). Suppose we turn the plane $\tilde{\omega}$ about P by replacing the straights ξ, η by other straights meeting at P. Each of these planes will have a curvature k. If all these k's are equal at P, we say that our space (5) has curvature k at this point. If k has the same value at all points P of our space, we say the space (5) is of constant curvature k.

For such spaces, we have the result Gauss established for $n=2$, that is, two n-way spaces of the same constant curvature have the same geometry (for not too large regions). Spaces of constant curvature have also this important property: in them, rigid bodies exist which may be moved about freely without any distortion. This of course is not generally true. For example, on an oval surface of revolution, like an egg, a small figure drawn on it can be moved without distortion only by revolving it about the axes of the surface. Riemann found that by a proper choice of variables the metric (5) of a space of constant curvature could be given the form

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{[1 + ek(x_1^2 + x_2^2 + \cdots + x_n^2)]^2}. \tag{16}$$

For $k=0$, and $n=3$, this reduces to (6); thus euclidean space is a space of zero curvature. If in (16) we take $e=-1$, $n=3$, we get the space of Lobachevsky and Bolyai. As the pseudosphere has a constant negative Gaussian curvature, the geometry on this surface is identical (for not too large regions) with plane Lobachevskian geometry, as first remarked by Beltrami (1868).

If we take $e=+1$, $n=2$, we have the geometry on a sphere. For $e=+1$, and $n=3$ we have a three-dimensional geometry never before imagined. As on the sphere, so here in three-way space, all straights are closed curves of constant length $\pi/\sqrt{k}$. Space is finite in extent, but without boundaries; its volume in fact is $\pi^2/(4k^{3/2})$.

This new space with positive k may be called spherical. As Klein first remarked, it has two forms; in the second form the straights are still closed, but of length $\pi/(2\sqrt{k})$, two points always determine uniquely a straight; whereas in the first form there is an exception, namely, when the two points are at a

distance $\pi/(2\sqrt{k})$. The connectivity of this second form of spherical space is quite complicated. For example, a watch moved along a straight l has been rotated through 180° about l when it returns to its point of departure. We have dwelt on spaces of constant curvature, partly because of their relation to the results obtained by projective methods, and partly because of their intrinsic value. But the geometry of spaces of constant curvature is only a small part of Riemannian geometry, as it is now called; that is, the geometry based on the general metric (5). In this geometry, the notions of euclidean space are extended so as to apply to this general metric. The reader who wishes to study this subject may consult one of the treatises mentioned in the footnote.*

4. *The Differential Method. Second Period. Non-Riemannian Geometry.* Riemannian geometry is based on a quadratic differential form. For a long time it was studied as an abstract science, just as the ancient Greeks studied the conic sections, or Gauss and Kummer studied the theory of numbers. Then all at once a great change came. In 1916 Einstein promulgated his general theory of relativity. The basis of this theory is a quadratic differential form (5) in four variables. This gave Riemannian geometry an enormous impetus. Riemannian geometry was no longer an abstract theory cultivated by a small body of mathematicians and quite neglected by the others; of a sudden it became of great practical value and widespread interest.

In 1917, Levi-Civita introduced the notion of infinitesimal parallel displacement. It was soon seen that this notion is more fundamental than that of the notion of distance ds between two nearby points, and that it can be used to find a geometry more general than that heretofore considered, a geometry now called non-Riemannian. It contains Riemannian geometry as a special case.

It is interesting to discover the origin of a notion so fundamental. Thompson and Tait in their *Natural Philosophy* (edition of 1879) discuss the kinematics of a surface rolling on a

* J. L. Coolidge, *The Elements of Non-Euclidean Geometry*, especially Chapters XV, XVI, Clarendon Press, 1909. L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926. W. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig, 1885.

plane without spin. They say in § 135: "The whole change of direction in a curved surface from one end to another of any arc of a curve traced on it is equal to the change of direction from end to end of the trace of this arc on a plane by pure rolling." In 1906 Brouwer hit upon the same notion for spaces of constant curvature. We may define the new notion as follows.

Suppose an elementary vector ξ whose components are $\xi_1, \xi_2, \cdots, \xi_n$ is moved along a curve K from a point x to the point $x+dx$, a distance ds as given by (5). If this vector changes its direction so that its components satisfy the n relations

$$\frac{d\xi_i}{ds} + \sum_{jl} C^i_{jl}\xi_l \frac{dx_j}{ds} = 0, \quad (i,j,k = 1,2, \cdots, n), \tag{17}$$

the C's being the Christoffel symbols (10) belonging to the metric (5), we say that ξ has received an infinitesimal parallel displacement along K. It is also said to be displaced geodetically along K. In fact the direction parameters $\xi_i = dx_i/ds$ of the tangent to a geodesic satisfy the equations (17) by virtue of (7). Hence the tangent to a geodesic receives an infinitesimal parallel displacement as it moves from x to $x+dx$ along the curve.

When ξ is moved geodetically from A to B along two different curves, we find the direction of ξ at B is not the same. If ξ is displaced geodetically along a closed curve G its change in direction after the circuit is given by the equation

$$\Delta\xi_i = \int_G \sum_{jk\lambda} \xi_\lambda R^i_{\lambda jk} dx_j dx_k, \tag{18}$$

where

$$R^i_{\lambda jk} = \frac{\partial C^i_{\lambda j}}{\partial x_k} - \frac{\partial C^i_{\lambda k}}{\partial x_j} + \sum_\alpha (C^i_{\alpha k} C^\alpha_{\lambda j} - C^i_{\alpha j} C^\alpha_{\lambda k}). \tag{19}$$

The relations (17), (18), (19) suppose that a metric (5) has been given us in advance. We may, however, proceed in another order. Let us suppose instead of starting with the quadratic form (5) and forming then the C^i_{ij} and the $R^i_{\lambda jk}$, we take n^3 functions of $x_1, x_2, \cdots, x_n$ which we will denote by Γ^i_{jl}, and

say that the vector ξ is displaced geodetically from a point x to $x+dx$ on a curve H when

$$d\xi_i + \sum_{jl} \Gamma^i_{jl} \xi_l dx_j = 0. \tag{20}$$

Let us define a geodesic to be such a curve that its tangent is displaced according to (20). When a vector ξ is displaced according to (20), that is, geodetically, around a closed circuit G, we find that its components receive the increments

$$\Delta\xi = \int_G \sum_{jkl} \xi_\lambda K^i_{\lambda jk} dx_j dx_k,$$

where the K's are obtained from (19) by replacing the C's by the new functions Γ^i_{jl}. With these Γ's and K's it is possible to develop a geometry in many ways analogous to that of Riemann. Still further generalizations have been made but these few remarks must suffice.*

In closing, the author expresses the hope that the present sketch may awaken a desire in the reader to acquaint himself more fully with this fascinating subject. At least it will give him an idea what tremendous strides non-euclidean geometry has made since Lobachevsky published his *Principles of Geometry* in 1829.

YALE UNIVERSITY

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* For further information the reader may consult the author's paper referred to in an earlier footnote, or L. P. Eisenhart, *Non-Riemannian Geometry*, Princeton University Press, 1928; D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, Berlin, 1922.

ON POLYNOMIAL SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY W. C. BRENKE

1. *Introduction.* Certain well known polynomials have a number of common properties. They arise as coefficients of t^n in the expansion of a generating function; they may be obtained by means of orthogonalization of a set of functions $x^n g(x)$ when the function $\rho(x)=g^2(x)$ and the interval are properly chosen; they may be regarded as polynomials which become orthogonal when multiplied by a proper factor $g(x)$; they satisfy a certain type of difference equation; they satisfy a certain type of differential equation. The results of this paper are based on the differential equation. Some of them are general statements of results already known for various classes of polynomials; others are believed to be new.

2. *Polynomial Solutions.* Consider the differential equation

$$P(x)y_n'' + Q(x)y_n' + \lambda_n R(x)y_n = 0, \tag{1}$$

where the coefficients P, Q, R are assumed to be real polynomials in the real variable x none of which vanish identically and λ_n is a polynomial in n. We assume that y_n is a polynomial of the form†

$$y_n = x^n + C_1^{(n)} x^{n-1} + C_2^{(n)} x^{n-2} + \cdots + C_n^{(n)}, \quad (n = 0,1,2, \cdots), \tag{2}$$

where the coefficient of the highest power of x is unity. On substituting $y_0=1$, $y_1=x+a_1$, $y_2=x^2+ax+b$, we find that P and Q must contain the factor R. Dividing out this factor and putting $\lambda_1=1$, as may be done without loss of generality, we find that the differential equation must have the form

* Presented to the Society, November 26, 1927, and December 1, 1928.

† When reference is made to various standard polynomials it will be assumed that they have been put into this form by use of a suitable factor.

$$py_n'' + qy_n' + \lambda_n y_n = 0, \tag{3}$$

where

$$p = \alpha x^2 + \beta x + \gamma, \quad q = -(x + a_1), \quad \lambda_n = n - n(n-1)\alpha, \tag{4}$$

the value of λ_n being uniquely determined when α is given.

For any assigned values of the constants $\alpha, \beta, \gamma, a_1$, the equation (3) will then give a complete system of polynomials starting with $y_0 = 1$ and $y_1 = x + a_1$, provided that the parameter values λ_n are all distinct. The values of the coefficients in (2) will then be

$$C_1^{(n)} = \frac{na_1 - n(n-1)\beta}{\lambda_n - \lambda_{n-1}}; \tag{2'}$$

$$C_t^{(n)} = \frac{(n-t+1)[a_1 - (n-t)\beta]\, C_{t-1}^{(n)}}{\lambda_n - \lambda_{n-t}} - \frac{(n-t+2)(n-t+1)C_{t-2}^{(n)}}{\lambda_n - \lambda_{n-t}}; \quad (t = 2, 3, \cdots n);$$

$$C_0^{(n)} = 0.$$

The case of equal parameter values will be considered under §9.

3. *Orthogonal Systems.* These polynomials will form an orthogonal system if the equation $p = 0$ has real and distinct roots $x = a$ and $x = b$, infinite roots not excluded, and if the constants $\alpha, \beta, \gamma, a_1$ are conditioned by certain inequalities. We shall assume $a < b$. To obtain these inequalities we first find, by the usual process, that

$$\int_a^b \rho y_n y_m dx = 0, \quad n \neq m,$$

provided that

$$\rho p = e^{\int (q/p)dx}, \text{ and } \rho p = 0 \text{ at } a \text{ and } b. \tag{5}$$

Then, if we assume that $\rho(x)$ is not negative in (a, b), the functions $\rho^{1/2} y_n$ form an orthogonal system for the interval (a, b), which may reach to infinity in either or both directions. This leads to the consideration of the following three cases.

CASE 1.

$$\alpha \neq 0; \quad p = -\alpha(x-a)(b-x);$$

$$a, b \text{ real and finite}; \qquad a < b.$$

The value of ρ from (5) will be

$$\rho = -\frac{1}{\alpha}(x-a)^{A-1}(b-x)^{B-1},$$

where

$$A = \frac{a+a_1}{\alpha(b-a)}, \quad B = -\frac{b+a_1}{\alpha(b-a)}.$$

The existence of $\int_a^b \rho dx$ will require the inequalities $A>0$, $B>0$, that is,

$$a+a_1>0 \text{ and } b+a_1<0 \text{ if } \alpha>0;$$
$$a+a_1<0 \text{ and } b+a_1>0 \text{ if } \alpha<0.$$

The case $\alpha>0$ leads to a contradiction since $a<b$; the other is equivalent to

$$\alpha<0; \quad -b<a_1<-a.$$

Then also $\rho p=0$ at a and b, and p and ρ are positive in (a, b). This gives the Jacobi polynomials.

CASE 2.

$$\alpha=0; \quad \beta\neq 0; \quad p=\beta\left(x+\frac{\gamma}{\beta}\right).$$

$\beta<0$ gives: $p(x)\geqq 0$ in $(a,b)=(-\infty, -\gamma/\beta)$;

$\beta>0$ gives: $p(x)\geqq 0$ in $(a,b)=(-\gamma/\beta,\infty)$.

Equation (5) now gives

$$\rho = \pm\frac{1}{\beta}e^{-x/\beta}(\pm x\pm\gamma/\beta)^{-(a_1-\gamma/\beta)/\beta-1},$$

where the upper or the lower signs are to be taken according as β is greater than or less than zero.

The existence of $\int_a^b \rho dx$ requires that $(a_1-\gamma/\beta)/\beta<0$, that is,

$$a_1 \lesseqgtr \gamma/\beta \text{ according as } \beta \gtrless 0.$$

Then ρp vanishes at a and b, and ρ and p are positive in (a, b). We obtain the generalized polynomials of Laguerre.

CASE 3.

$$\alpha=\beta=0; \quad p=\gamma\neq 0; \quad (a,b)=(-\infty,\infty).$$

Here

$$\rho = \frac{1}{\gamma} e^{-(x+\alpha_1)^2/(2\gamma)}.$$

The existence of $\int_a^b \rho dx$ requires $\gamma > 0$, which also makes ρp vanish at a and b, and $\rho > 0$ in (a, b). This gives the general Hermite polynomials.

As examples we may consider the differential equations

$$(x \pm 1)y_n'' - xy_n' + ny_n = 0.$$

Polynomial solutions are:

(With the plus sign)	(With the minus sign)
$y_0 = 1,$	$y_0 = 1,$
$y_1 = x,$	$y_1 = x,$
$y_2 = x^2 - 2x - 1,$	$y_2 = x^2 - 2x + 1,$
$y_3 = x^3 - 6x^2 + 3x + 4,$	$y_3 = x^3 - 6x^2 + 9x - 4,$
. . . .	. . . .

The first set forms an orthogonal system, the second does not.

4. *Normalizing Constants.* The polynomials obtained in §3 form an orthogonal system and hence satisfy a difference equation of the form*

$$y_n = (a_n + x)y_{n-1} - b_n y_{n-2}, \qquad (n > 1). \tag{6}$$

Here

$$b_n = c_{n-2}^2/c_{n-1}^2, \qquad a_n = -c_{n-1}^2 k_{n-1}; \tag{7}$$

$$1/c_n^2 = \int_a^b \rho y_n^2 dx, \quad k_n = \int_a^b \rho x y_n^2 dx. \tag{8}$$

The constants c_n are the normalizing constants for the polynomials y_n, so that, if $f_n = \rho^{1/2} c_n y_n$, then

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* J. Shohat, *Sur le développement de l'intégrale · · · et sur les polynomes de Tchebycheff*, Rendiconti di Palermo, vol. 47 (1923), pp. 25–46. The case (a, b) finite has also been treated by G. Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres*, Journal de Mathématiques, (3), vol. 4 (1878); and by N. Abramescu, *Sulle serie di polinomi di una variable complessa. Le serie di Darboux*, Annali di Matematica, (3), vol. 31 (1922), pp. 207–249.

$$\int_a^b f_n f_m dx = 1 \text{ or } 0,$$

according as n and m are equal or unequal.

Direct substitution of (2) in (6) shows that a_n and b_n may also be obtained by means of the equations

$$(9) \qquad a_n = C_1^{(n)} - C_1^{(n-1)}; \quad b_n = a_n C_1^{(n-1)} - C_2^{(n)} + C_2^{(n-1)}.$$

When the constants b_n are known, the normalizing constants are given by the formula

$$(10) \qquad 1/c_n^2 = \left[\prod_2^{n+1} b_r \right] \int_a^b \rho dx.$$

5. *Determinant Form of the Difference Equation.* The solution of the difference equation (6) may be written in determinant form as

$$(11) \quad y_n = \begin{vmatrix} a_1 + x & b_2 & 0 & \cdots & 0 \\ 1 & a_2 + x & b_3 & \cdots & 0 \\ 0 & 1 & a_3 + x & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdots & a_{n-1} + x & b_n \\ 0 & 0 & \cdots & 1 & a_n + x \end{vmatrix}.$$

This is seen directly when the determinant is expanded according to the elements of the last row or column.

If we denote this determinant by $D(a, b, 1, x)$, we may also write it in the equivalent form $D(a, b^{1/2}, b^{1/2}, x)$, which is formed by replacing the elements immediately above and immediately to the left of each principal diagonal element $a_{r,r}$ by $b_r^{1/2}$, $r = 2, \cdots, n$. This is a symmetric secular determinant with real elements, since the constants b_r are positive by (7). From this follows again that the roots of y_n are all real.*

* Kowalewski, *Determinantentheorie*, 1909, p. 126.

6. *Representation by Derivatives.* The orthogonal polynomials y_n are representable as derivatives of order n by the formula*

$$h_n y_n = (1/\rho) D_x^n (\rho p^n), \tag{12}$$

where h_n is the coefficient of x^n on the right.

If, in the expression

$$L(y) \equiv p y_n'' + q y_n' + \lambda_n y_n,$$

where the coefficients are as in (4), we substitute $u_n = \rho y_n$, we obtain

$$M(u) \equiv p u_n'' + (2p' - q) u_n' + (\lambda_n + p'' - q') u_n.$$

We now form the expression

$$N(v) \equiv p v_n'' - [(n-2)p' + q] v_n' + \mu_n v_n,$$

where

$$\mu_n = \lambda_n + \tfrac{1}{2}(n-1)(n-2)p'' + (n-1)q' = 1 - 2(n-1)\alpha.$$

Differentiation of $N(v)$ n times gives $M(v)$. Also $v_n = \rho p^n$ gives

$$N(\rho p^n) = \rho p^n [\mu_n + q' + (n-1)p''] = 0.$$

Therefore the expression on the right of (12) is a solution of $L(y) = 0$. But this expression is readily shown to be a polynomial of degree n by use of the relation $d(\rho p)/dx = \rho q$. Hence it differs from y_n at most by a constant factor.

7. *Generating Function.* The orthogonal polynomials y_n have a generating function

$$\phi(x,t) = \frac{\rho(y)}{\rho(x)} \frac{\partial y}{\partial x}, \quad \text{where} \quad y = x + t p(y).$$

The Lagrange expansion formula gives the formal result

$$\rho(y) \frac{\partial y}{\partial x} = \rho(x) + \rho(x) \sum_1^\infty D_x^n (\rho p^n) t^n / n!.$$

Division by $\rho(x)$ and taking account of (12) gives

$$\phi(x,t) = \sum_0^\infty h_n y_n t^n / n!.$$

* See Abramescu, loc. cit., pp. 213–14, 217.

EXAMPLE. We apply these results to the Laguerre polynomials considered by G. Szegö.* They are regarded as arising by the orthogonalization of

$$x^{n+k}g(x) \equiv e^{-x/2}x^{n+k}; \quad k \geqq 0; \quad (a,b) = (0, \infty).$$

Then $p(x)=\beta x$; also, $d(\rho p)/dx=\rho q$ gives $q=(2k+1-x)/\beta$. Taking $\beta=1$, we have the differential equation in our standard form

$$xy_n'' + (2k + 1 - x)y_n' + ny_n = 0.$$

This gives $C_1^{(n)}$ and $C_2^{(n)}$, then a_n and b_n; hence the difference equation is

$$y_n = (1 - 2k - 2n + x)y_{n-1} - (n - 1)(n + 2k - 1)y_{n-2}.$$

The normalizing constants are

$$1/c_n^2 = \prod_{m=2}^{n+1}(m - 1)(m + 2k - 1)\int_0^\infty e^{-x}x^{2k}dx$$
$$= \prod_{m=2}^{n+1}(m - 1)(m + 2k - 1)\Gamma(2k + 1).$$

The generating function is

$$\frac{\rho(v)}{\rho(x)}\frac{\partial y}{\partial x}, \quad (y = x + ty,\ \rho(x) = e^{-x}x^{2k})$$

which gives at once

$$\phi(x,t) = (1 - t)^{-2k-1}e^{-xt/(1-t)}.$$

8. *The Case* $Q(x)\equiv 0$ *in* (1). The considerations in §2 show that we must have $\lambda_1=0$.

Then the differential equation must be

$$py_n'' + \lambda_n y_n = 0, \tag{3'}$$

where

$$p = \alpha x^2 + \beta x + \gamma = -\alpha(x - a)(b - x),\ \alpha \neq 0;$$
$$\lambda_n = -n(n - 1)\alpha.$$

Hence we may write

* *Ein Beitrag zur Theorie der Polynome von Laguerre und Jacobi*, Mathematische Zeitschrift, vol. 1 (1918).

$$(3'') \qquad (x-a)(b-x)y_n'' + n(n-1)y_n = 0.$$

Then $\rho = 1/p$, and $\int_a^b \rho dx$ does not exist. From (3''), y_n vanishes at a and b, $n>1$, and therefore

$$\int_a^b \frac{1}{p} y_n y_m dx = (y_n y_m' - y_m y_n')_a^b = 0, \qquad n \neq m\ ;\ n,m > 1.$$

Hence the polynomials form an incompletely orthogonal system. The normalizing constants are given by

$$1/c_n^2 = \int_a^b (y_n^2/p)dx, \qquad n > 1,$$

and c_0 and c_1 do not exist. Other results of the preceding sections require corresponding restrictions.

9. *Exceptional Cases.* The exceptional case $\lambda_n - \lambda_{n-t} = 0$; $n>1$, $t=2, 3, \cdots, n$.

The value of λ_n from (4) gives

$$\lambda_n - \lambda_{n-t} = \alpha t(1/\alpha + 1 + t - 2n) = 0.$$

Therefore $1/\alpha$ must be a positive integer. It follows that in general, solutions y_n will not exist as polynomials for the following values of n:

when $1/\alpha = 2m$, for $n = m+1, m+2, \cdots, 2m+1$,

when $1/\alpha = 2m-1$, for $n = m+1, m+2, \cdots, 2m$,

$(m = 1, 2, 3, \cdots)$.

(Exceptions consist of cases where individual constants $C_t^{(n)}$ become indeterminate and may be given arbitrary values.) The differential equation furnishes an incomplete system of polynomials, which, since $\alpha > 0$, will not be orthogonal.

For example, the equation

$$(x^2/4 - 1)y_n'' - (x+1)y_n' + [n(5-n)/4]y_n = 0$$

has the solutions $y_0 = 1$, $y_1 = x+1$, $y_2 = x^2+4x+4$; y_3, y_4, and y_5 do not exist as polynomials of degree 3, 4, 5, respectively; then we again have polynomial solutions

$$y_6 = x^6 - 4x^5 - 4x^4 + 32x^3 - 16x^2 - 64x + 64,$$

and so on.

THE UNIVERSITY OF NEBRASKA

A PROPERTY OF CONTINUA EQUIVALENT TO LOCAL CONNECTIVITY*

BY W. A. WILSON

1. *Introduction.* It is the purpose of this note to develop a property of continuous curves which seems to have been overlooked by the numerous writers on this subject and which, in effect, gives a new definition of this extensive class of continua. The property in question is given by the following definition.

A continuum (or space) M is said to be divisible if, for every pair of sub-continua A and B without common points, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.

It will be shown that for a compact, metric, and connected space the concepts of divisibility and local connectivity at every point are equivalent. It will also be shown that this is true for any continuum, bounded or unbounded, located in a euclidean space.

As a preliminary we note that we cannot replace the word "sub-continua" in the above definition by the word "points." To see the truth of this statement consider the plane continuum M consisting of a segment ab of length 1 and an infinite set of arcs of radii n $(n=2, 3, \cdots)$, each subtended by ab and of length less than a semi-circumference. It is readily seen that this continuum is not divisible in the sense of the above definition, but that for any two *points* A and B we can decompose M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.

2. THEOREM. *Let M be a metric, separable, connected space which is locally connected at each point. Then M is divisible.*

PROOF. Let A and B be any proper sub-continua of M and $A \cdot B = 0$. Let x be any point of A, let $\epsilon > 0$ be less than one-third the distance between x and B, and let the symbol $V_\epsilon(x)$ denote the set of points of M whose distance from x is less than

* Presented to the Society, August 30, 1929.

ϵ. Since M is locally connected at each point, there is a $\delta>0$ and a sub-continuum C_x of M such that $V_\delta(x) \subset C_x \subset V_\epsilon(x)$. Let F be the union of the continua $\{C_x\}$ as x ranges over A plus the limiting points of this set. Then F is a continuum, every point of A is an inner point of F, and $B \cdot F = 0$. Only the last statement requires explanation. If B contains a point a of F, it lies on no C_x, but is the limit of a sequence $\{a_n\}$ of points such that each a_n lies on some C_n corresponding to a point x_n of A. Let ϵ_n be the ϵ corresponding to x_n. These quantities can have no positive lower bound, since each ϵ_n is less than one-third the distance from x_n to B. But this means that for a partial sequence $\epsilon_n \to 0$, whence $x_n \to a$, contrary to the hypothesis that $A \cdot B = 0$. Thus $B \cdot F = 0$.

In like manner it is easy to see that there is a sub-continuum G of M such that every point of B is an inner point of G and $A \cdot G = 0$. Since M is everywhere locally connected, each component of $M-(F+G)$ has limiting points on $F+G$. Set P' equal to the union of F and all the components of $M-(F+G)$ having limiting points on F, and define Q' corresponding in like manner to G. Then $M = P'+Q'$. Also $P = \overline{P}'$ and $Q = \overline{Q}'$ are continua and, since A and B are inner sets of F and G, respectively, $P \cdot B = Q \cdot A = 0$. Thus M is divisible.

Note. Similar reasoning shows that, if $A_1, A_2, \cdots, A_n$ are proper sub-continua of M and $A_i \cdot A_j = 0$ if $i \neq j$, then M can be decomposed into n continua $\{P_i\}$ such that for each i, we have $A_i \subset P_i$ and $P_i \cdot [\sum_1^n A_i - A_i] = 0$.

3. THEOREM. *Let M be a compact, metric, and connected space which is divisible. Then M is locally connected at each point.*

PROOF. Let x be any point of M, let $\epsilon>0$, and let K be any component of $M-V_\epsilon(x)$. Since M is divisible, $M = P_k + Q_k$, where P_k and Q_k are continua and $K \cdot P_k = x \cdot Q_k = 0$. Now $R_k = M - P_k$ is a region and $M - V_\epsilon(x)$ is contained in the union of the regions $\{R_k\}$.

Since M is compact, and $M - V_\epsilon(x)$ is closed, the Borel theorem applies and a finite set of the regions $\{R_k\}$, say R_1, $R_2, \cdots, R_n$, covers $M - V_\epsilon(x)$. As each R_i is a sub-set of a corresponding continuum Q_i which does not contain x, we have shown that $M - V_\epsilon(x)$ is contained in n sub-continua of M, none of which contains x. Since n is finite, it follows that for

each $\epsilon>0$ there is a least integer m such that $M-V_\epsilon(x)\subset \sum_1^m S_i$, where each S_i is a continuum, $x\cdot S_i=0$, and $S_i\cdot S_j=0$ if $i\neq j$.

Let $R=M-\sum_1^m S_i$. Then R is a region and $R\subset V_\epsilon(x)$. Let ρ be any component of R not containing x and μ be the component containing x. No component ρ has limiting points on more than one S_i. For, if such a ρ existed, $x\cdot\bar{\rho}=0$ and $S_i+\bar{\rho}+S_j$ would be a continuum not containing x, contrary to the assumption that $M-V_\epsilon(x)$ is contained in not less than m such continua. On the other hand, each component of R has limiting points on at least one S_i. Since $\overline{R}$ is a compact closed set, no two continua $\{S_i\}$ have common points, and $M=\overline{R}+\sum_1^m S_i$ is a continuum, it follows that $\overline{R}$ contains a continuum H irreducible between S_1 and $\sum_2^m S_i$. Then $H'=H-H\cdot\sum_1^m S_i$ is connected and lies on a component of R. As $\overline{H'}$ contains points of more than one S_i, $H'\subset\mu$. Hence $\bar{\mu}\cdot S_1\neq 0$, and likewise $\bar{\mu}\cdot S_i\neq 0$ for every i.

If, for every $\epsilon>0$, there is some $\delta>0$ such that $\mu\supset V_\delta(x)$, the theorem is proved. For $\bar{\mu}$ is a continuum of diameter less than or equal to 2ϵ, which makes the oscillation of M about x not greater than 2ϵ. As ϵ can be chosen at pleasure, the oscillation is zero.

The alternative possibility is that for some $\epsilon>0$ and every $\delta>0$, $V_\delta(x)$ contains a point of some ρ. We show that this leads to a contradiction as follows. If this case occurs, every $V_\delta(x)$ must contain an infinite sequence of points $\{a_j\}$ converging to x, where each a_j lies on a component ρ_j of R and every $\bar{\rho}_j$ has points on precisely one S_i, say S_1. (This last statement is valid since m is finite and no component of R except μ has limiting points on more than one S_i.)

Since M is divisible, there is a decomposition $M=N_1+N_2$, where N_1 and N_2 are continua and $N_1\cdot S_1=N_2\cdot x=0$. Then N_1 contains an infinity of the points $\{a_j\}$, because x is a limiting point of the set $\{a_j\}$. Since $x\cdot N_2=0$ and $S_1\subset N_2$, we have $N_2\cdot S_i=0$ for $i\geqq 2$. For S_1 can be joined to no other S_i by a continuum not containing x.

Let L be a sub-continuum of N_1 irreducible between some a_j and $x+\sum_2^m S_i$, and let $L'=L-L\cdot(a_j+x+\sum_2^m S_i)$. Now L' is connected, $L'\subset M-\sum_1^m S_i$, and either $x\subset L'$ or $\overline{L'}\cdot S_i\neq 0$ for some $i\geqq 2$. In the first case $L'+x+a_j$ is connected and

hence $a_j \subset \mu$, a contradiction. In the second case the connected set $L'+a_j \subset \rho_j$ and $\bar{p}_j \cdot S_i \neq 0$ for $i=1$ and some other i, another contradiction. Thus the assumption that there is no δ such that $V_\delta(x) \subset \mu$ is false and the theorem is proved.

4. THEOREM. *Let M be a bounded or unbounded continuum lying in a euclidean space. For M to be locally connected at every point it is necessary and sufficient that, for every pair of sub-continua A and B of M, there is a decomposition of M into two continua P and Q such that $P \cdot B = Q \cdot A = 0$.*

PROOF. The condition is necessary by §2. If M is bounded it is sufficient by §3. Let us assume then that M is unbounded. The proof for this case is obtained by suitably modifying that of §3 to take care of the fact that M is not compact.

Regarding M as a space, proceed as in the first paragraph of the proof of §3. Let F be the frontier of $V_\epsilon(x)$. As F is compact, it is covered by a finite set of the continua $\{Q_k\}$, say $Q_1, Q_2, \cdots, Q_n$. Then $T = M - V_\epsilon(x) + \sum_1^n Q_i$ is a closed set and a finite number of its components cover F. Let these be $C_1, C_2, \cdots, C_k$. It is easily seen that $T = \sum_1^k T_i$, where each T_i is a closed set, $T_i \cdot T_j = 0$ if $i \neq j$, and $C_i \subset T_i$ for each i. If T_1 is not a continuum, $T_1 = U_1 + W_1$, where $U_1 \cdot W_1 = 0$ and U_1 and W_1 are closed; suppose that $C_1 \subset U_1$. Then M is the sum of two closed sets, W_1 and $V_\epsilon(x) + U_1 + \sum_2^m T_i$, without common points, —an impossibility as M is a continuum. Thus $C_1 = T_1$ and likewise $C_i = T_i$ for every i. We have then, as in the second paragraph of the proof of §3, for each ϵ a least integer m such that $M - V_\epsilon(x)$ is covered by m mutually exclusive sub-continua of M, which we may denote by $\{S_i\}$ and none of which contains x.

The remainder of the proof of §3 needs no alteration for this case. The existence of the irreducible continuum L in the last paragraph can be established in the same way as the existence of H in the third paragraph.

YALE UNIVERSITY

THE SIMPLEST INVOLUTORIAL TRANSFORMATION CONTAINED MULTIPLY IN A LINE COMPLEX*

BY VIRGIL SNYDER

Involutorial birational transformations contained multiply in a linear line complex are interesting, as they furnish the simplest examples of involutions that are probably irrational. The following example is the simplest possible case of such a transformation; it possesses features that are characteristic, and may itself be irrational.†

Let $(0, 0, z_3, z_4)$ be a variable point on the line $x_1=0$, $x_2=0$, and $\lambda_1 x_3^2+\lambda_2 x_4^2=0$ an involution I_2 of pairs of planes of a pencil, the axis being skew to the first line. Let us suppose that the point (z) and the planes (λ) are connected by the relation $z_3\lambda_1^2+z_4\lambda_2^2=0$. A point (y) of space determines the pair of planes $x_3^2y_4^2-x_4^2y_3^2=0$ of the pencil; hence $z_3=y_3^4$, $z_4=y_4^4$. A point (x) on the line joining (y) to (z) has coordinates of the form $\rho x_1=\sigma y_1$, $\rho x_2=\sigma y_2$, $\rho x_3=\sigma y_3+\tau z_3$, $\rho x_4=\sigma y_4+\tau z_4$. This line meets the plane conjugate to that determined by (y) in (y'), corresponding to $\sigma=y_3^3+y_4^3$, $\tau=-2$.

The points (y), (y') are therefore associated in an involutorial birational transformation, the equations of which have the form

$$(I_4)\qquad \begin{cases} \rho x_1' = (x_3^3 + x_4^3)x_1, \\ \rho x_2' = (x_3^3 + x_4^3)x_2, \\ \rho x_3' = (x_4^3 - x_3^3)x_3, \\ \rho x_4' = (x_3^3 - x_4^3)x_4. \end{cases}$$

The transformation I_4 is contained doubly in the special linear line complex, the axis of which is $x_1=0$, $x_2=0$ in the sense that each line of the complex contains two pairs of conjugate points in I_4. Every plane through the axis is transformed into

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† Another involution of order 2 that is probably irrational is described in the Giornale di Matematiche, vol. 61 (1923), pp. 125–128.

itself, but the axis is not in the base of the web of surfaces conjugate to the planes of space.

In the correspondence $z_4y_3^4-z_3y_4^4=0$ between the points (z) of the axis and the planes (y) of the pencil through the fundamental line, there are five positions of the point such that a corresponding plane passes through it. Of these, two are the double planes of the quadratic involution I_2 which correspond to $z_3=0$, $z_4=0$. At these points all four associated planes coincide. The points in $y_3=0$ and in $y_4=0$ are paired in the harmonic homology having its center on the axis $x_1=0$, $x_2=0$ and its axis as the fundamental line. The two centers (0, 0, 1, 0) and (0, 0, 0, 1) are fixed points in I_4.

The other three points on the axis which contain an associated plane are (0, 0, 1, 1), (0, 0, ω, 1), (0, 0, ω^2, 1), $\omega^3=1$. The respective planes of I_2 not passing through the points are $x_3+x_4=0$, $x_3+\omega x_4=0$, $x_3+\omega^2x_4=0$. The three points are isolated fundamental points of the first order, and their images are the corresponding planes. The relation between the elements at each of the fundamental points and the associated image plane is perspectivity. Every plane through the axis is transformed into itself.

The line $x_3=0$, $x_4=0$ is fundamental of order 3. Any point $(y_1, y_2, 0, 0)$ on it has for images the three lines joining the point to the three fundamental points on the axis. The locus of these lines is the triad of planes

$$x_3 - x_4 = 0,\ x_3 - \omega x_4 = 0,\ x_3 - \omega^2 x_4 = 0.$$

Moreover, every point on the line $x_3=0$, $x_4=0$ is invariant. The pencil of planes through $x_3=0$, $x_4=0$ is invariant in I_4, but the individual planes are interchanged as in the quadratic involution I_2. The jacobian of the system consists of the six planes $x_3^6-x_4^6=0$ through the fundamental line, each counted twice.

The conjugate of an arbitrary line of the complex is a plane quartic curve C_4 lying in the plane determined by the line and the axis, meeting the given line in two pairs of conjugate points. These curves form a Jonquières net $|C_4|$, having a common triple point at the intersection of the plane with the fundamental line, with all three tangents common. The other three basis points are the three fundamental points on the axis. The

Jonquières involution is not perspective. Each point of an arbitrary line of the plane, regarded as a point (y), is associated with a point (z) on the axis, and the point of intersection is not self-corresponding, hence the line (yz) which also contains (y') envelopes a conic.

But when (y) describes a line skew to the axis, although each point (y) uniquely determines a point (z) on the axis, the same point (z) is associated with four points (y), hence the locus (yz) is now a ruled surface of order 5, having (z) as a four-fold and the given line as simple directrix. This surface contains C_4, the conjugate of the given line, and the curve described by the other pair of points on each line (yz). If $(ax)=0$, $(bx)=0$ define the given line, the image lies on the quadric

$$(a_1x_1 + a_2x_2)(b_3x_3 - b_4x_4) - (b_1x_1 + b_2x_2)(a_3x_3 - a_4x_4) = 0,$$

which does not contain the given line unless $a_3b_4 - a_4b_3 = 0$, that is, the line meets the axis. But in that case the quadric is composite,

$$(b_3x_3 - b_4x_4)(a_1x_1 + a_2x_2 + kb_1x_1 + kb_2x_2) = 0,$$

and the image C_4 lies in the plane through the axis.

The conjugate of an arbitrary plane is a quartic ruled surface having the fundamental line for triple directrix. It also passes through the three fundamental points on the axis, and at each has for tangent planes those connecting these points with the fundamental line, hence two different surfaces, conjugates of two planes, touch each other along all three sheets through the fundamental line. The proper curve of intersection, of order 4, meets the axis in the three fundamental points. The fundamental line counts for order 12 in the intersection of the two surfaces.

The conjugate surface contains the three lines joining the point of intersection of the given plane with the fundamental line to the three fundamental points on the axis. If a plane passes through a fundamental point on the axis, its conjugate is composite, having the associated fundamental plane as component. The linear line congruence having the axis and the fundamental line for directrices is invariant in I_4.

The involution I_4 is of genus 0. The curve of intersection of any plane with its conjugate surface is composite, consisting

of four concurrent lines. The plane $\sum a_i x_i = 0$ is transformed into

$$(x_3^3 + x_4^3)(a_1x_1 + a_2x_2) + (x_4^3 - x_3^3)(a_3x_3 - a_4x_4) = 0,$$

hence the curve of intersection is defined by

$$a_3x_3^4 + a_4x_4^4 = 0, \qquad \sum a_i x_i = 0.$$

These four lines are arranged in two pairs, in perspective from the point in which the given plane meets the axis.

As the plane $\sum a_i x_i = 0$ turns about any fixed line $\sum b_i x_i = 0$ in it, this composite curve generates a quintic ruled surface R_5 having the fundamental line for four-fold directrix. If the Plücker coordinates of the given line are (a_{jk}), the equation of the surface is

$$a_{13}x_1x_3^4 + a_{23}x_2x_3^4 + a_{14}x_1x_4^4 + a_{24}x_2x_4^4 + a_{34}x_3x_4(x_3^3 - x_4^3) = 0.$$

There is an ∞^4 system of such surfaces. Those through any point (y) also contain (y') and the other pair of conjugate points on the line (yy'). This ∞^3 subsystem is of the form

$$\begin{aligned} &A\{y_3y_4(y_3^3 - y_4^3)x_1x_3^4 - y_1y_3^4x_3x_4(x_3^3 - x_4^3)\} \\ &+ B\{y_3y_4(y_3^3 - y_4^3)x_2x_3^4 - y_2y_3^4x_3x_4(x_3^3 - x_4^3)\} \\ &+ C\{y_3y_4(y_3^3 - y_4^3)x_1x_4^4 - y_1y_4^4x_3x_4(x_3^3 - x_4^3)\} \\ &+ D\{y_3y_4(y_3^3 - y_4^3)x_2x_4^4 - y_2y_4^4x_3x_4(x_3^3 - x_4^3)\} = 0. \end{aligned}$$

If this system is mapped on the points of another space S_3' the representation establishes a (1, 4) correspondence between the points of the two spaces S_3', S_3; all the points common to any triad of associated surfaces lie on a line, which belongs to the complex $p_{12} = 0$. Moreover, all the surfaces pass through four fixed points on the line (yy').

Consider the surfaces $T_5(r)$, $T_5(s)$ of this system associated with two skew lines r, s, neither belonging to the complex $p_{12} = 0$. Their residual intersection lies on the regulus defined by r, s and the axis k. It is of type (4, 5) on the quadric containing the regulus. The two points in which the fundamental line $x_3 = 0$, $x_4 = 0$ meets the quadric $H_2(r, s, k)$ are four-fold points on C_9. A general C_9 of type (4, 5) has 16 apparent double points hence is of genus 12. But this C_9 has two four-fold points with coplanar branches, hence it is of genus zero.

The curves of this system which pass through the fixed points on the line (yy') are the images of the straight lines of the space S_3'; all four image points in S_3 of a point of the line in S_3' lie on the curve. The correspondence contains two coincidences. Each curve of the subsystem is invariant in I_4.

Since C_9 has three simple points on the axis, the images of which are three straight lines, and has two four-fold points on the fundamental line, the image of each of which is three straight lines, the residual is the curve C_9 of type (4, 5) having the same fundamental points; that is, C_9 is invariant in I_4.

If the system $|R_5|$ is mapped projectively on the hyperplanes of S_4 the result is a quadric variety, hence the double involution is rational, but this is not sufficient to conclude that I_4 is rational. If the more general system $|s_1+s_4|$: $I^3 3P$ is mapped on the hyperplanes of an auxiliary space, much higher dimensions must be employed.

The product of I_4 by the central harmonic homology $(0,0,0,1)$, $x_4=0$, is a non-involutorial T_4, which leaves every line of a linear congruence invariant. The plane $x_4=0$ is composed of invariant points. Any ruled surface belonging to the congruence is invariant under an infinite discontinuous group of birational transformations. The simplest case of k pairs is that of the form $\rho x_1' = (x_3^{2k-1}+x_4^{2k-1})x_1$, etc. There are now $2k-1$ fundamental points on the axis, each having the other plane of the pair in I_2 for conjugate. The fundamental line $x_3=0$, $x_4=0$ is of multiplicity $2k-1$, its image surface consisting of a plane to each fundamental point on the axis.

A plane not containing the axis nor the fundamental line meets its conjugate surface in k pairs of lines, all belonging to a pencil. When $k=1$, a known quadratic transformation results.

The product of the transformation I_{2k} and the axial harmonic homology H: $x_1'=x_1$, $x_2'=x_2$, $x_3'=-x_3$, $x_4'=-x_4$ is a new transformation $S=I_{2k}H=HI_{2k}$, which is contained to multiplicity $k-1$ in the same special linear complex, and has two planes $x_3=0$, $x_4=0$ of invariant points.

CORNELL UNIVERSITY

ON RELATED DIFFERENCE AND DIFFERENTIAL SYSTEMS†

BY W. M. WHYBURN

In a recently published article‡ I considered the system of differential equations

$$(1) \qquad dy_i/dx = \sum_{j=1}^{j=m} A_{ij}(x)y_j + \beta_i(x), \qquad (i = 1, \cdots, m),$$

where $A_{ij}(x)$, $\beta_i(x)$, $(i, j=1, \cdots, m)$ are summable, real functions of the real variable x on $X: a \leqq x \leqq b$. I proved that for one set of definitions of the coefficients of the difference system

$$(2) \quad \Delta^* y_i(r)/\Delta x(r) = \sum_{j=1}^{j=m} {}^*A_{ij}(r)\,{}^*y_j(r) + {}^*\beta_i(r), \ (i = 1, \cdots, m),$$

on E_n: $x_{0n}=a$, $x_{1n}, \cdots, x_{nn}=b$, where the asterisk indicates a function defined on E_n (replacing the bold-face type in my former paper), ${}^*f(r)={}^*f(x_{rn})$ and $\Delta^*f(r)={}^*f(r+1)-{}^*f(r)$, *every* solution of this system goes over in the limit as n, the number of points in E_n, becomes infinite, in such a way that X is *completely*§ subdivided to the corresponding solution, that is, the solution having the same initial values at $x=a$, of the differential system (1). The present paper shows that the conclusions of our former paper, stated above, are valid for *all possible* methods of defining the coefficients of system (2), so long as $\lim_{n\to\infty} {}^*A_{ij}(p)=A_{ij}(x)$, $\lim_{n\to\infty} {}^*\beta_i(p)=\beta_i(x)$, almost everywhere on X, and there exists a summable function $G(x)$ on X such that $|{}^*A_{ij}(p)|$, $|{}^*\beta_i(p)| < G(x)$ for all n, $(i, j=1, \cdots, m)$, on $I_{pn}: x_{pn} \leqq x \leqq x_{p+1,n}$, where p varies with n in such a way that the point x belongs to I_{pn}. It shows further that the approach to the limit is *uniform* on X and that all of these conclusions are valid for any law of complete subdivision of X by the points of E_n. Our former paper indicated ready adaptations of the work to non-

† Presented to the Society, November 29, 1929.

‡ American Journal of Mathematics, vol. 51 (1929), pp. 265–286.

§ By a *complete* subdivision is meant one such that $\lim n\to\infty$ maximum $\Delta x(i)=0$.

linear systems and to systems containing parameters and the work of the present paper can be similarly adapted. Finally, an examination of the proofs together with the demonstrated uniformity of approach immediately yields estimates of the degree of the approximation of the solutions of the differential system by those of the difference system. The generality of the systems considered and the conclusions obtained cause the present paper to have an important bearing on several recent papers† and to contain some of the results of these papers as special cases.

THEOREM. *Let a method of complete subdivision of X be given and let* ${}^*A_{ij}(r)$, ${}^*\beta_i(r)$, $(i,j=1,\cdots,m;\ r=0,1,\cdots,n)$, *be defined for each n so that* $\lim_{n\to\infty} {}^*A_{ij}(p)=A_{ij}(x)$, $\lim_{n\to\infty} {}^*\beta_i(p)=\beta_i(x)$, *almost everywhere on X and so that* $|{}^*A_{ij}(p)|$, $|{}^*\beta_i(p)|<G(x)$ *on* I_{pn}, $(p=0,\cdots,n-1)$, *where $G(x)$ is summable on X. Let* ${}^*y_i(p)$, $(i=1,\cdots,m)$, *be any solution of* (2) *and let* ${}^*y_i(0)=\alpha_i$; *then if* $y_1(x),\cdots,y_m(x)$ *is the solution of* (1) *such that* $y_i(a)=\alpha_i$, $(i=1,\cdots,m)$, *we have* $\lim_{n\to\infty} {}^*y_i(p)=y_i(x)$ *uniformly on X.*

PROOF. Examine §1, pages 276–282, of my former paper.‡ Using the notation of that paper, we establish two lemmas:

LEMMA 1. $\mathrm{Lim}_{k\to\infty} {}^*y_{kj}(p)={}^*y_j(p)$, $(j=1,\cdots,m)$, *uniformly with respect to p and n.*

LEMMA 2. $\mathrm{Lim}_{n\to\infty} {}^*z_{kj}(p+1)=z_{kj}(x)$, $(j=1,\cdots,m)$, *uniformly on X.*

PROOF OF LEMMA 1. Let ${}^*u(r)={}^*g(r)\Delta x(r)$, where ${}^*g(r)$ is the lower bound of $G(x)$ on I_{rn} (existent since $G(x)>0$) and let $M=m[\alpha+\int_a^b G(x)dx]$, where $\alpha=\sum_{i=1}^m|\alpha_i|$. Repetition of the proof of page 278 of paper J shows that, for each k,

$$|{}^*z_{kj}(p)|<(m^kM/k!)\left[\sum_{r=0}^{n-1}u(r)\right]^k\leqq D_k,$$

† See Plancherel, Bulletin des Sciences Mathématiques, (2), vol. 47 (1923), pp. 153–160, 170–177.

Bogoliouboff and Kryloff, Annals of Mathematics, (2), vol. 29 (1928), p. 255ff. Rose Whelan, this Bulletin, vol. 35 (1929), pp. 105–122.

‡ Whyburn, loc. cit. This paper will be referred to as paper J. It is to be noted that bold-faced letters in paper J are replaced by ordinary letters that are preceded by asterisks in the present paper.

for all n and p, where

$$D_k = (m^k M/k!)\left[\int_a^b G(x)dx\right]^k.$$

Since D_k is independent of n and p and since the series of constants $\sum_{k=1}^{k=\infty} D_k$ is absolutely convergent, it follows that

$$\lim_{k\to\infty} {}^*y_{kj}(p) = \lim_{k\to\infty} \sum_{r=0}^{k} {}^*z_{rj}(p) = {}^*y_j(p)$$

uniformly with respect to p and n.

PROOF OF LEMMA 2. We use mathematical induction. By the notation ${}^*f(x)$, we mean a function that has the value ${}^*f(p)$ on $x_{pn} \leqq x < x_{p+1,n}$. (This notation replaces the previous notation $f^{(p)}(x)$ of paper J.) For $k=0$, we have

$$|\,{}^*z_{0j}(p+1) - z_{0j}(x)\,| = \left|\int_a^x [{}^*\beta_j(t) - \beta_j(t)]dt\right.$$

(3)

$$\left. + \int_x^{x_{p+1,n}} {}^*\beta_j(t)dt\right| \leqq \int_a^x |\,{}^*\beta_j(t) - \beta_j(t)\,|\,dt + \int_x^{x_{p+1,n}} G(t)dt,$$

for every x on X. Let $\epsilon > 0$ be assigned; then from the absolute continuity of $\int_x^{x_{p+1,n}} G(t)dt$ and from the fact that the subdivision of X is complete, we have that for every $n \geqq N_1$ and all x on X, the last integral of (3) is less then $\epsilon/2$. The limit as n becomes infinite of the integrand in the first integral of (3) is zero almost everywhere on X and this integrand is bounded by the summable function $2G(x)$ for all x on X and for all n. An application of Lebesgue's theorem† shows that the limit as n becomes infinite of this integral is zero. Hence for all $n > N_2$,

$$\int_a^b |\,{}^*\beta_j(t) - \beta_j(t)\,|dt < \epsilon/2.$$

It follows from (3) that for all $n > N \geqq N_1, N_2$ and for all x on X, $|\,{}^*z_{0j}(p+1) - z_{0j}(x)\,| < \epsilon$. This establishes the lemma for $k=0$. We show that its validity for $k=r-1$ implies its validity for $k=r$. The expression ${}^*z_{rj}(p+1) - z_{rj}(x)$ is made up of m terms of which

$$\sum_{i=0}^{p} {}^*A_{js}(i)\,{}^*z_{r-1,\,s}(i)\Delta x(i) - \int_a^x A_{js}(t) z_{r-1,s}(t)dt$$

is typical. Let $\epsilon > 0$ be given; we show that each of the m terms of the foregoing type is uniformly less than ϵ/m on X for all $n > N$. Write such a term as $I_1 - I_2 + I_3$, where

† *Leçons sur l'Intégration*, etc., Paris, 1904, p. 114.

$$I_1 = \int_a^x A_{js}(t)\,[{}^*z_{r-1,s}(t) - z_{r-1,s}(t)]\,dt,$$

$$I_2 = \int_a^x {}^*z_{r-1,s}(t)\,[{}^*A_{js}(t) - A_{js}(t)]\,dt,$$

$$I_3 = \int_x^{x_{p+1,n}} {}^*A_{js}(t)\,{}^*z_{r-1,s}(t)\,dt.$$

From the validity of the lemma for $k=r-1$, we can choose N_{1s} such that for all $n>N_{1s}$, $|{}^*z_{r-1,s}(t)-z_{r-1,s}(t)|<\epsilon/[3m\int_a^b G(t)dt]$ on X. Hence for all $n>N_{1s}$, $|I_1|<\epsilon/(3m)$ for every x on X. Now for all x on X,

$$|I_2| < D_{r-1}\int_a^b |{}^*A_{js}(t) - A_{js}(t)|\,dt,$$

and an application of Lebesgue's theorem† to the integral on the left-hand side of this inequality shows that for all $n>N_{2s}$ this integral is less than $\epsilon/(3mD_{r-1})$ and hence $|I_2|$ is less than $\epsilon/(3m)$ for every x on X. From the absolute continuity of

$$\int_x^{x_{p+1,n}} D_k G(t)\,dt$$

and from the fact that the subdivision of X is complete, we get N_{3s} such that for all $n>N_{3s}$, $|I_3|<\epsilon/(3m)$ for every x on X. Combining the results for I_1, I_2, I_3, and taking into account the m terms that occur when $s=1, 2, \cdots, m$, we get

$$|{}^*z_{rj}(p+1) - z_{rj}(x)| < \epsilon,\ n > N \geqq N_{11}, N_{21}, N_{31}, N_{12}, \cdots, N_{3m},$$

and for every x on X. This establishes Lemma 2.

PROOF OF THEOREM. An argument‡ entirely analogous to that used in proving Lemma 1 shows that $\lim_{k\to\infty} y_{kj}(x) = y_j(x)$ uniformly on X. Hence if ϵ is an assigned positive number there exists K_1 so that for $k>K_1$, $|y_{kj}(x)-y_j(x)|<\epsilon/3$ on X. In accordance with Lemma 1, let K be chosen greater than K_1 and so that for all n and p,

(4) $$|{}^*y_{Kj}(p) - {}^*y_j(p)| < \epsilon/3, \quad \text{for all } n \text{ and } p,$$

(5) $$|y_{Kj}(x) - y_j(x)| < \epsilon/3, \quad \text{for every } x \text{ on } X.$$

† Loc. cit.

‡ This is but the ordinary argument for successive approximations.

By Lemma 2 choose N so that for all $n > N$,

$$(6) \qquad |\, {}^*z_{rj}(p) - z_{rj}(x)\,| < \frac{\epsilon}{3(K+1)}, \qquad (r = 0, \cdots, K),$$

and all x on X. From

$$ {}^*y_{Kj}(p) = \sum_{r=0}^{K} {}^*z_{rj}(p), \qquad y_{Kj}(x) = \sum_{r=0}^{K} z_{rj}(x),$$

and (6), follows

$$(7) \qquad |\, {}^*y_{Kj}(p) - y_{Kj}(x)\,| < \epsilon/3, \quad \text{for } n > N \text{ and all } x \text{ on } X.$$

A combination of (4), (5), and (7) yields

$$|\, y_j(x) - {}^*y_j(p)\,| < \epsilon, \qquad \text{for all } n > N \text{ and for every } x \text{ on } X.$$

This establishes the desired uniformity and completes the proof of the theorem.

Obviously one need but check through the foregoing work and that of paper J to obtain estimates of the degree of approximation.†

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† The approximations obtained in this way can of course be improved when more special systems are considered.

A LEMMA OF THE THEORY OF LINEAR DIFFERENTIAL SYSTEMS*

BY J. D. TAMARKIN

The reading of the preceding note by W. M. Whyburn suggests to me a lemma which, although it is simple, deserves to be stated and proved separately. The results of Whyburn† as well as those of Miss Whelan‡ may be derived immediately from this lemma.

We shall use the matrix notation and designate by German capitals the square matrices of n rows and n columns, and by small German letters the n-dimensional vectors, with the usual agreements as to the products of matrices by matrices or by vectors and so on.

By a solution of a linear differential system of nth order with integrable (L) coefficients we shall mean a vector whose components are absolutely continuous and satisfy the system almost everywhere.

LEMMA. *Let the coefficients* $a'_{ij}(x)$, $a''_{ij}(x)$, $b'_i(x)$, $b''_i(x)$ *of the linear differential system*

$$\frac{d\mathfrak{y}'(x)}{dx} = \mathfrak{A}'(x)\cdot\mathfrak{y}'(x) + \mathfrak{b}'(x), \tag{1}$$

$$\frac{d\mathfrak{y}''(x)}{dx} = \mathfrak{A}''(x)\cdot\mathfrak{y}''(x) + \mathfrak{b}''(x) \tag{2}$$

be integrable on $(0, 1)$. *Let*

$$\max_i \left\{ \sum_{j=1}^{n} \int_0^1 | a'_{ij}(x) | \, dx \right\} = M_1,$$

$$\max_i \left\{ \sum_{j=1}^{n} \int_0^1 | a''_{ij}(x) | \, dx \right\} = M_2,$$

* Presented to the Society, October 26, 1929.

† See also § 4 of Whyburn's paper referred to, on p. 94 of this issue.

‡ This Bulletin, vol. 35 (1929), pp. 112–119. Our lemma is but a slight modification of and is proved in a much the same way as lemma v (p. 116) of Miss Whelan.

$$\max_i \left\{ \int_0^1 | b_i'(x) | \, dx \right\} = N_1,$$

$$\max_i \left\{ \int_0^1 | b_i''(x) | \, dx \right\} = N_2,$$

$$\max_i \left\{ \int_0^1 \sum_{j=1}^n | a_{ij}'(x) - a_{ij}''(x) | \, dx \right\} = \epsilon,$$

$$\max_i \left\{ \int_0^1 | b_i'(x) - b_i''(x) | \, dx \right\} = \eta.$$

$$(i = 1, 2, \cdots, n),$$

Let $\mathfrak{c}$ *be a vector with constant components and* $\max_i \{ |c_i| \} = C$. *Then, if* $\mathfrak{y}'(x)$, $\mathfrak{y}''(x)$ *are respectively the solutions of* (1) *and* (2) *satisfying the initial conditions*

$$\mathfrak{y}'(0) = \mathfrak{y}''(0) = \mathfrak{c}, \tag{3}$$

we have on (0, 1)

$$| y_i'(x) - y_i''(x) | \leqq \begin{cases} (\eta + \epsilon K_1) e^{M_2}, & K_1 = (C + N_1) e^{M_1}, \\ (\eta + \epsilon K_2) e^{M_1}, & K_2 = (C + N_2) e^{M_2}, \end{cases} \tag{4}$$

$$(i = 1, 2, \cdots, n).$$

We observe first that the solution of a system

$$\frac{d\mathfrak{y}(x)}{dx} = \mathfrak{A}(x) \cdot \mathfrak{y}(x) + \mathfrak{b}(x) \tag{5}$$

with integrable coefficients exists and is uniquely determined by its initial values:

$$\mathfrak{y}(0) = \mathfrak{c}. \tag{6}$$

Furthermore, on setting

$$\max_i \left\{ \sum_{j=1}^n \int_0^1 | a_{ij}(x) | \, dx \right\} = M, \quad \max_i \left\{ \int_0^1 | b_i(x) | \, dx \right\} = N,$$

the following inequalities hold:

$$| y_i(x) | \leqq (C + N) e^M, \qquad (i = 1, 2, \cdots, n). \tag{7}$$

The proof of these facts is readily obtained by means of the classical method of successive approximations in a slightly

modified form, due to the non-boundedness of the functions $a_{ij}(x)$ and $b_i(x)$. The method of successive approximations must be applied not directly to the system (5) but rather to the equivalent system of integral equations*

$$\mathfrak{y}(x) = \mathfrak{c} + \int_0^x \mathfrak{b}(t)dt + \int_0^x \mathfrak{A}(t)\cdot\mathfrak{y}(t)dt. \tag{8}$$

Now we can replace systems (1) and (2) with the initial conditions (3) by the equivalent systems of integral equations

$$\mathfrak{y}'(x) = \mathfrak{c} + \int_0^x \mathfrak{b}'(t)dt + \int_0^x \mathfrak{A}'(t)\cdot\mathfrak{y}'(t)dt,$$

$$\mathfrak{y}''(x) = \mathfrak{c} + \int_0^x \mathfrak{b}''(t)dt + \int_0^x \mathfrak{A}''(t)\cdot\mathfrak{y}''(t)dt,$$

whence

$$\mathfrak{y}'(x) - \mathfrak{y}''(x) \equiv \mathfrak{y}(x) = \int_0^x \mathfrak{b}(t)dt + \int_0^x \mathfrak{A}(t)\mathfrak{y}(t)dt$$

where we have put

$$\mathfrak{b}(x) = \mathfrak{b}'(x) - \mathfrak{b}''(x) + \begin{cases} \int_0^x [\mathfrak{A}'(t) - \mathfrak{A}''(t)]\cdot\mathfrak{y}'(t)dt \\ \text{or} \\ \int_0^x [\mathfrak{A}'(t) - \mathfrak{A}''(t)]\cdot\mathfrak{y}''(t)dt \end{cases}$$

and

$$\mathfrak{A}(x) = \begin{cases} \mathfrak{A}''(x) \text{ or} \\ \mathfrak{A}'(x). \end{cases}$$

Using (7) we get the proof of our lemma.

If inequality (7) is assumed our lemma can be derived from a general existence theorem by Carathéodory.† But this inequality does not follow directly from Carathéodory's discussion. Moreover the method used above presents the advantage of giving an explicit estimate for the difference.

We can leave to the reader the derivation of the results of W. M. Whyburn and of Miss Whelan, as well as the applica-

* Tamarkin, *Some general problems of the theory of ordinary linear differential equations*, Petrograd, 1917, pp. 29–39 (in Russian).

† *Vorlesungen über reelle Funktionen*, Teubner, Leipzig, 1927, p. 678.

tion of this lemma to the case where the coefficients of the system (5) and the initial data (6) depend on a parameter. The interval (0, 1) of course can be replaced by an arbitrary interval (a, b), the initial values $\mathfrak{y}'(0)$, $\mathfrak{y}''(0)$ may be made distinct and the whole theory can be extended to systems of infinitely many equations, under suitable restrictions upon the matrices and vectors involved.

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ON THE NUMBER OF APPARENT MULTIPLE POINTS OF VARIETIES IN HYPERSPACE

BY B. C. WONG

By an apparent point of multiplicity s on a variety in r-space we mean a line which passes through a given point in r-space and meets the variety in s distinct points. In order that the number of such apparent s-fold points on a variety V_x^m of order m and of x dimensions be finite, we must have $r=st+1$, $x=(s-1)t$, where t is the number less one of the hypersurfaces intersecting in the variety. In other words, the number of apparent s-fold points on a $V_{(s-1)t}^m$ which is the intersection of $t+1$ hypersurfaces in S_{st+1} is finite. It is the purpose of this paper to determine this number and also to determine its upper and lower limits.

We shall use the symbols $H_s^{(r)}$, $\bar{h}_s^{(r)}$ to denote respectively the maximum and minimum number of apparent s-fold points that a $V_{(s-1)t}^m$ of any order m in $S_r[r=st+1]$ can have, and the symbol $h_s^{(r)}$ to denote the number of those that a $V_{(s-1)t}^m$ of order $m=n_1n_2\cdots n_{t+1}$, which is the complete intersection of $t+1$ hypersurfaces of orders $n_1, n_2, \cdots, n_{t+1}$ respectively, ordinarily has. Thus if $s=2$, $t=1$ and therefore $r=3$, a curve C^m in S_3 can have at most

$$H_2^{(3)} = (m-1)(m-2)/2\,, \tag{1}$$

and at least

$$\bar{h}_2^{(3)} = m(m-2)/4, \qquad \text{for } m \text{ even}, \tag{2}$$

and

$$\bar{h}_2^{(3)} = (m-1)^2/4, \qquad \text{for } m \text{ odd}, \tag{2'}$$

apparent double points. If C^m is the complete intersection of two surfaces of orders n_1, n_2 respectively, the number of apparent double points it has is

$$h_2^{(3)} = n_1 n_2 (n_1 - 1)(n_2 - 1)/2. \tag{3}$$

To illustrate further, let $s = r-1$, $t = 1$. The number of apparent $(r-1)$-fold points on the complete intersection $V_{r-2}^{n_1 n_2}$ of two hypersurfaces of orders n_1, n_2 respectively in S_r is known to be*

$$h_{r-1}^{(r)} = (r-1)! \binom{n_1}{r-1} \binom{n_2}{r-1}. \tag{4}$$

We shall show later that the maximum number of such points on a V_{r-2}^m of any order m is

$$H_{r-1}^{(r)} = \binom{m-r+2}{r-1}, \tag{5}$$

and we shall show that the minimum number on a V_{r-2}^m of order $m = (r-1)m' + k$, where k may be any one of the integers from 0 to $r-2$, is

$$\bar{h}_{r-1}^{(r)} = (r-1)! \binom{m'}{r-1} + k(r-2)! \binom{m'}{r-2}. \tag{6}$$

We are to derive the formulas for $H_s^{(r)}$, $\bar{h}_s^{(r)}$, and $h_s^{(r)}$. For this purpose we make use of a method analogous to the one employed by the writer in the derivation of the formula for $h_{r-1}^{(r)}$ above in the paper referred to. We proceed to determine the formula for $h_s^{(r)}$ first.

Let the variety $V_{(s-1)t}^m [m = n_1 n_2 \cdots n_{t+1}]$ be the complete intersection of $t+1$ hypersurfaces of orders $n_1, n_2, \cdots, n_{t+1}$ respectively in $S_r [r = st+1]$. Consider for a moment the hypersurfaces as being composed entirely of hyperplanes: $a_1^{(1)}, a_2^{(1)}, \cdots, a_{n_1}^{(1)}; a_1^{(2)}, a_2^{(2)}, \cdots, a_{n_2}^{(2)}; \cdots; a_1^{(t+1)}, a_2^{(t+1)}, \cdots, a_{n_{t+1}}^{(t+1)}$. Then $V_{(s-1)t}^m$ is composed of $n_1 n_2 \cdots n_{t+1}$ $(s-1)t$-spaces

$$a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_{t+1}}^{(t+1)}, \qquad [i_j = 1, 2, \cdots, n_j].$$

We write $(x_1, x_2, \cdots, x_{t+1})$ in place of $a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_{t+1}}^{(t+1)} \cdots$ and

* B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35, pp. 339–343.

regard $x_1, x_2, \cdots, x_{t+1}$ as the cartesian coordinates of a point in a $(t+1)$-space. Removing the restriction that the hypersurfaces be made up of hyperplanes, we take the totality of the $m=n_1n_2\cdots n_{t+1}$ points $(x_1, x_2, \cdots, x_{t+1})$ for all the integral values of x_j from 1 to $n_j[j=1, 2, \cdots, t+1]$ to be the symbolic representation of $V^m_{(s-1)t}$, proper or improper. In this representation any point or any set of values of $(x_1, x_2, \cdots, x_{t+1})$ taken alone means an $S_{(s-1)t}$ forming a part of $V^m_{(s-1)t}$ and, in general, if we divide the total group of the m points in any way into two or more groups, then these groups represent two or more varieties into which $V^m_{(s-1)t}$ degenerates. Any group of s points with all of the $t+1$ corresponding coordinates chosen different, as $(1, 1, \cdots, 1)$, $(2, 2, \cdots, 2), \cdots, (s, s, \cdots, s)$ or $(1, 2, 3, \cdots, t+1)$, $(2, 3, 4, \cdots, 1)$, $(3, 4, 5, \cdots, 2), \cdots, (s, s+1, s+2, \cdots, s-1)$ represents s $(s-1)t$-spaces which are not concurrent. Since only one line can be drawn passing through a given point and meeting each of s $(s-1)t$-spaces in general positions, the existence of such a group of s points in the symbolic representation of $V^m_{(s-1)t}$ means the existence of an apparent s-fold point on $V^m_{(s-1)t}$. The total number of such groups is the number $h_s^{(r)}$ we are seeking.

To derive the formula for $h_s^{(r)}$ we first consider a group G of s^{t+1} points $(x_1, x_2, \cdots, x_{t+1})$ in the representation of $V^m_{(s-1)t}$ whose coordinates take on all the possible sets of values formed from any s positive integers not exceeding $n_j[j=1, 2, \cdots, t+1]$ with repetitions allowed. Thus one such group G for $s=2$, $t=2$ consists of the following 8 points $(1, 1, 1)$, $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$, $(2, 2, 1)$, $(2, 1, 2)$, $(1, 2, 2)$, $(2, 2, 2)$. Then the question is, How many groups of s points all of whose corresponding coordinates are different are there contained in a group G of s^{t+1}? The answer is obviously $N=(s!)^t$. Thus for $s=t=2$, the number of pairs of points with corresponding coordinates different is 4 and they are $(1, 1, 1)$, $(2, 2, 2)$; $(2, 1, 1)$, $(1, 2, 2)$; $(2, 2, 1)$, $(1, 1, 2)$; $(2, 1, 2)$, $(1, 2, 1)$.

Next we ask the question: How many groups of the type G, each consisting of s^{t+1} points, as defined above, are contained in the total representation of $V^m_{(s-1)t}$? Since each x_j may take on all the integral values from 1 to n_j, the total number N' of G is $\binom{n_1}{s}\binom{n_2}{s}\cdots\binom{n_{t+1}}{s}$ as each G involves all s of the integral values.

Hence the total number of groups of s points with all corresponding coordinates unlike is the product NN', that is,

$$h_s^{(r)} = (s!)^t \binom{n_1}{s}\binom{n_2}{s}\cdots\binom{n_{t+1}}{s} \tag{7}$$

which is the required formula. If $s=2, t=1$, or if $s=r-1, t=1$, we have formulas (3) or (4) above, respectively.

Now to determine $H_s^{(r)}$, that is, the greatest number of apparent s-fold points a $V^m_{(s-1)t}$ can have. To give concreteness to the reasoning, we proceed to derive formula (5) for $H_{r-1}^{(r)}$. In this case $s=r-1$, $t=1$. The variety V^m_{r-2} must be the partial intersection of two hypersurfaces and the m points (x_1, x_2) in the symbolic representation of the variety must be such that every two consecutive points have one of the coordinates take on values alike and the other take on different values. One such variety may be represented by the following m points: (1, 1), (1, 2), (2, 2), (2, 3), $\cdots$, $(m/2, (m+2)/2)$ if m is even or (1, 1), (1, 2), (2, 2), (2, 3), $\cdots$, $([m+1]/2, [m+1]/2)$ if m is odd. In either case, the total number of groups of $r-1$ points whose corresponding coordinates are all different is

$$H_{r-1}^{(r)} = \binom{m-r+2}{r-1},$$

which is formula (5) above. If $r=3$, this is reduced to formula (1).

To find $H_s^{(r)}$ we reason in an analogous way. The variety $V^m_{(s-1)t}$ is the partial intersection of $t+1$ hypersurfaces. The representation of this variety consists of m points such that every group of $q[q=2, 3, \cdots, s+1]$ points has $q-1$ corresponding coordinates take on different values. For example, if $t=3$, $V^m_{3(s-1)}$ may be represented by the following m points: (1, 1, 1, 1), (2, 1, 1, 1), (2, 2, 1, 1), (2, 2, 2, 1), (2, 2, 2, 2), (3, 2, 2, 2), (3, 3, 2, 2), etc. It can be easily verified that the total number of groups of s points with the corresponding coordinates all different is

$$H_s^{(r)} = \binom{m-st+t}{s}. \tag{8}$$

This formula is reduced to (5) if $s=r-1$, $t=1$ and to (1) if $s=2$, $t=1$.

To find $\bar{h}_s^{(r)}$, we notice that, if the number of apparent s-fold points on a $V^m_{(s-1)t}$ is to be minimum, the variety must be the complete or a partial intersection of $t+1$ hypersurfaces, t of which are of order s.* Putting $n_1=n_2=\cdots=n_t=s$, $n_{t+1}=m/s^t$ in formula (7) above, we have

$$\bar{h}_s^{(r)}=(s!)^t\binom{m/s^t}{s}, \tag{9}$$

for m a multiple of s^t. If m is not a multiple of s^t, let $m=s^tm'+k$ where k may be zero or any of the integral values from 1 to s^t-1. Substituting in (7), we have

$$(s!)^t\binom{(m-k)/s^t}{s} \quad \text{or} \quad (s!)^t\binom{m'}{s},$$

to which is to be added

$$k[(s-1)!]^t\binom{(m-k)/s^t}{s-1} \quad \text{or} \quad k[(s-1)!]^t\binom{m'}{s-1},$$

which is the number of groups of s points with the $t+1$ corresponding coordinates different, each group containing one and only one of the k points in the symbolic representation of $V^m_{(s-1)t}$. Hence

$$\left\{\begin{aligned}\bar{h}^{(r)} &= (s!)^t\binom{(m-k)/s^t}{s}+k[(s-1)!]^t\binom{(m-k)/s^t}{s-1}\\ &= (s!)^t\binom{m'}{s}+k[(s-1)!]^t\binom{m'}{s-1},\end{aligned}\right. \tag{10}$$

where $m'=(m-k)/s^t$. If $k=0$, this is reduced to (9). If $s=2$, $t=1$, we have (2) and if $s=r-1$, $t=1$, we have (6).

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* If any of the hypersurfaces is of order less than s, then $\bar{h}_s^{(r)}=0$.

ON A DECOMPOSITION OF A QUATERNARY QUADRATIC FORM*

BY G. Y. RAINICH

For a binary quadratic form it is very easy to prove that it can be decomposed in a unique way into a sum of a form of the type $e(x^2+y^2)$ and another which by an orthogonal transformation can be reduced to $q(x^2-y^2)$.† It is the purpose of this note to prove an analogous theorem for quadratic quaternary forms, namely, that *such a form can be presented as the sum of three forms*:

(a) *a form of the type* $e(x^2+y^2+z^2+t^2)$,

(b) *a form which by an orthogonal transformation can be reduced to* $q(x^2+y^2-z^2-t^2)$, *and*

(c) *a square of a linear form* $(ax+by+cz+dt)^2$.

The decomposition in general is unique; an exceptional case is indicated below.

It seems that the decomposition is interesting in itself but it gains in interest because of possible application in physics. Translated into tensor language the theorem may be stated as follows: a symmetric tensor of rank two in four-dimensional space may be presented, in general in a unique way, as the sum of a tensor of the type of a hydrodynamical tensor $\delta g_{ij}+\rho u_i u_j$, where δ corresponds to pressure, ρ to density and u_i is a unit vector, the four-dimensional velocity; and an electromagnetic stress-energy tensor.‡ If the theorem were true in pseudo-euclidean space, it would mean that the contracted Riemann tensor, set equal to the sum of the two above tensors, determines (in general, uniquely) the field of matter and the electromagnetic field. However, the situation in pseudo-euclidean space is more complicated and the theorem there is not true without modifica-

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† These two types correspond, if the forms are interpreted as second differential forms of surfaces, to a sphere and to a minimal surface respectively.

‡ For a proof that this tensor is of the type (b), see Proceedings of the National Academy of Science, vol. 10 (1924), p. 126; or Transactions of this Society, vol. 27 (1925), p. 117.

tion and additional restrictions which lie in the nature of things; we limit ourselves here to the euclidean case and leave the applications aside, reserving the treatment of the pseudo-euclidean case for another publication.*

By an orthogonal transformation every given form can be reduced to one of the type

$$fx^2 + gy^2 + hz^2 + kt^2. \tag{1}$$

The problem can be stated then in the following way: is it possible to subtract from (1) an expression of the type (a) and an expression of the type (c) so that the remainder should be reducible by an orthogonal transformation to the type (b)? It is convenient, instead of quadratic forms to consider corresponding matrices. Without writing out separately the matrices corresponding to (1), (a), and (c), we write at once the matrix corresponding to the first minus the other two; it is:

$$\begin{Vmatrix} f - a^2 - e & -ab & -ac & -ad \\ -ba & g - b^2 - e & -bc & -bd \\ -ca & -cb & h - c^2 - e & -cd \\ -da & -db & -dc & k - d^2 - e \end{Vmatrix}. \tag{2}$$

This matrix must be transformable by an orthogonal transformation into one of the type

$$\begin{Vmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -q \end{Vmatrix}. \tag{3}$$

The square of this matrix is q^2 times the unit matrix. But an orthogonal transformation brings a unit matrix into a unit matrix; therefore the square of the matrix (2) must be a unit matrix multiplied by a number, and the elements outside of the main diagonal of this matrix (the square of (2)) must necessarily be zero; we obtain thus six equations:

* Compare L. Hanni, *Verwendung des Energie-Impulstensors*, etc., Tôhoku Mathematical Journal, vol. 27 (1926), pp. 277–292.

$$
\begin{aligned}
&ab(a^2+b^2+c^2+d^2+2e-f-g)=0,\\
&ac(a^2+b^2+c^2+d^2+2e-f-h)=0,\\
&ad(a^2+b^2+c^2+d^2+2e-f-k)=0,\\
&bc(a^2+b^2+c^2+d^2+2e-g-h)=0,\\
&bd(a^2+b^2+c^2+d^2+2e-g-k)=0,\\
&cd(a^2+b^2+c^2+d^2+2e-h-k)=0.
\end{aligned}
\tag{4}
$$

We could satisfy these equations by making three of the four numbers a, b, c, d zero, for example, $a=b=c=0$; then the matrix (2) becomes a diagonal matrix without any transformation, but in order that it should be possible to make it assume the form (3) by the choice of the two numbers e and d still at our disposal, two of the given numbers f,g,h must be equal. Unless two of the numbers f,g,h,k are equal we will not achieve our purpose by making three of the numbers a,b,c,d vanish. We may try to make two of these numbers zero, for example,

$$a=b=0. \tag{5}$$

This satisfies the first five of the equations (4) and to satisfy the last one we have to make

$$c^2+d^2+2e=h+k. \tag{6}$$

The matrix (6) becomes now

$$
\left\|\begin{matrix}
f-e & 0 & 0 & 0\\
0 & g-e & 0 & 0\\
0 & 0 & h-c^2-e & -cd\\
0 & 0 & -dc & k-d^2-e
\end{matrix}\right\|. \tag{7}
$$

An orthogonal transformation of the variables z and t only is required to transform this into a diagonal matrix, and any additional orthogonal transformations that would not destroy diagonality would not affect the elements $f-e$ and $g-e$. The other two elements of the resulting diagonal matrix are the roots of the quadratic equation

$$x^2-(h-c^2-e+k-d^2-e)x+\begin{vmatrix} h-c^2-e & -cd\\ -dc & k-d^2-e\end{vmatrix}=0;$$

the coefficient of x in this equation is zero in view of (6) so that these two elements must be identified with q and $-q$ of the matrix (3), and therefore the elements $f-e$ and $g-e$ also must be two numbers whose sum is zero; we obtain in this way $f-e+g-e=0$, or

$$e = \tfrac{1}{2}(f+g). \tag{8}$$

It only remains for us to arrange it so that the two opposite numbers $f-e$ and $g-e$ be of the same absolute value as the roots of the quadratic equation above, that is, that the products of these two pairs be the same, if we want to have a matrix of the form (3). We have thus the equation

$$(h-c^2-e)(k-d^2-e)-c^2d^2=(f-e)(g-e)$$

which in view of (6) and (8) becomes

$$kc^2+hd^2=hk-fg. \tag{9}$$

Eliminating e from (6) and (8) we obtain also

$$c^2+d^2=h+k-(f+g). \tag{10}$$

The last two equations can be easily solved for c^2 and d^2, if we assume that $h\neq k$, and give

$$d^2=\frac{(h-f)(h-g)}{h-k}, \qquad c^2=\frac{(k-f)(k-g)}{k-h}. \tag{11}$$

This furnishes real values for c and d only if the fractions in the right hand sides of (11) are both positive; but then we have a choice of six possibilities as to which two of the four numbers a,b,c,d we set equal to zero and among these six possibilities there is always one and only one which gives real values for the remaining two. To see this it is enough to notice that for d, for example, to be real it is necessary and sufficient that the corresponding number h be such that

$$(h-f)(h-g)(h-k)$$

be positive; this means that h must be either the greatest or the second smallest among the numbers f,g,h,k. We must therefore arrange our notations in such a way that h and k be the greatest and second smallest among the numbers e,f,g,h; then

the formulas (11), (8), and (5) furnish us a solution of the problem.

We still have to consider the possibility of satisfying the equations (4) by making vanish more than one of the expressions in brackets in the left hand sides of these equations and less than two of the numbers a,b,c,d; but it is easy to see that this is possible only in the exceptional case mentioned above, namely, where two of the numbers f,g,h,k are equal. Barring this exceptional case we have then a unique system of values for e, a^2, b^2, c^2, and d^2; but we can choose the signs of c and d.

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ON THE PHRAGMÉN-BROUWER THEOREM*

BY W. A. WILSON

1. *Introduction.* The purpose of this note is to give an elementary demonstration of the validity of the Phragmén-Brouwer theorem in spaces satisfying certain conditions. This theorem has been proved and generalized by Urysohn and Alexandroff† for cartesian spaces by means of the theory of dimensionality, but the great importance of the theorem seems to the writer to justify the offering of another proof which involves only the elementary principles of the point-aggregate theory. As is well known, to prove this theorem for a space is the same as demonstrating that, if m and n are any two points of the space and C is an irreducible cut between m and n, then C is a continuum. It is this form which will be used and the proof is deduced from the validity of the theorem in the euclidean plane.

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* Presented to the Society, October 26, 1929. Shortly after the submission of the manuscript of this note to the editors, C. Kuratowski published another proof of this theorem in the Fundamenta Mathematicae, vol. 14, pp. 304–310. This paper was not withdrawn, because the great importance of the theorem seemed to the author to warrant the belief that another demonstration would not be void of interest to readers.

† P. Alexandroff, *Sur les multiplicités cantoriennes et le théorème de Phragmén-Brouwer généralisé*, Comptes Rendus, vol. 183, pp. 722–724.

2. *Notation.* In addition to the familiar notations of the point-aggregate theory, the following will be used.

If a and b are two points, the distance between them is denoted by Dist (a,b).

The symbol $V_\epsilon(a)$ denotes the set of points whose distances from a are less than ϵ.

If C is a closed cut of space between the points (or point sets) A and B, we write that C is an $S(A,B)$.

The symbol $\overline{\lim}\, A_i$ denotes the upper closed limiting set of the sequence of sets $\{A_i\}$ as $i \to \infty$.

3. *Convex Spaces.* Let a and b be points of a metric space Z and let $k = \text{Dist } (a,b)$. For each value of t, $0 \leqq t \leqq k$, let there be one and only one point x such that Dist $(a,x) = t$ and Dist $(x,b) = k - t$. If this property pertains to each pair of points of Z, we say that Z is *convex.** The set of points $\{x\}$ is called a *linear segment,* or simply a *segment,* and is denoted by ab. Thus a convex space is one in which each pair of points determines a unique segment.

Every euclidean space is convex, but not conversely. As an example, consider a space consisting of three ordinary linear segments ax, bx, and cx lying in a euclidean plane. Let the distance between a pair of points on any one of these be measured as usual. If u, v, w are points on ax, bx, and cx, respectively, let Dist (u,v) be defined as Dist (u,x) + Dist (v,x), etc. In this space any two points can be joined by a unique segment, but we notice that two different segments, as uv and uw, may have a segment, here ux, in common.

The following properties are readily proved for any convex space in which every bounded set has at least one limiting point:

(a) it is a continuum;

(b) it is locally connected at each point;

(c) if $\{x_i\}$ and $\{y_i\}$ are sequences of points converging to x and y, respectively, Dist $(x_i, y_i) \to$ Dist (x, y) and $x_i y_i \to xy$, that is, the set xy is both the upper and the lower closed limiting set of the sequence of sets $\{x_i y_i\}$.

A space of this kind obviously includes euclidean space of any number of dimensions as a special case.

* This term is used in a slightly more restricted sense by Hausdorff; see *Mengenlehre,* p. 96.

4. THEOREM. *Let Z be a convex space in which every bounded infinite set has at least one limiting point. If m and n are two points and C is an irreducible $S(m,n)$, then C is a continuum.*

PROOF. We assume the opposite and arrive at a contradiction. Let $C=A+B$, where A and B are closed and $A\cdot B=0$. Let M and N be the components of $Z-C$ containing m and n, respectively. Since Z is everywhere locally connected, M and N are connected regions and C is the frontier of each of them.

Let h be a point of A and k one of B, and let $\epsilon>0$ be less than one third of the smaller of the quantities Dist (h,B) and Dist (k,A). Then $B\cdot V_\epsilon(h)=0=A\cdot V_\epsilon(k)$. Now there is a point a in M and a point c in N such that the segment $ac\subset V_\epsilon(h)$. Likewise there is a point b in M and a point d in N such that the segment $bd\subset V_\epsilon(k)$. Then $ac\cdot B$, $bd\cdot A$, and $ac\cdot bd$ are void. As C is an $S(m,n)$, we have $ac\cdot A\neq 0$ and $bd\cdot B\neq 0$. Since Z is everywhere locally connected, there are simple arcs ab and cd in M and N, respectively. Finally, there is no loss in generality in assuming that $(ab+cd)\cdot(ac+db)=a+b+c+d$. For $ac\cdot ab$ and $ac\cdot cd$ are closed sets; hence ac contains a segment $a'c'$ such that $a'c'\cdot ab=a'$ and $a'c'\cdot cd=c'$. Likewise, bd contains a segment $b'd'$ such that $b'd'\cdot ab=b'$ and $b'd'\cdot cd=d'$. Let $a'b'$ be the sub-arc of ab whose ends are a' and b', and $c'd'$ have a similar relation to cd. Clearly $a'b'$, $a'c'$, $b'd'$, and $c'd'$ have the properties assigned to ab, ac, bd, and cd.

In a cartesian plane let S be the closed square whose frontier is the union of the segments pq, pr, qs, and rs, each of unit length. Set up a homeomorphism between ab and pq, and between cd and rs, so that a, b, c, and d correspond to p, q, r, and s, respectively. Now let u be any point of pq and v one of rs so that uv is parallel to pr. In Z let x and y be the points of ab and cd corresponding to u and v, respectively, and let xy be the segment joining them. (If $u=p$, $xy=ac$; if $u=q$, $xy=bd$.) Let Q be the union of the sets $\{xy\}$ for all positions of u on pq. Set up a homeomorphism between each xy and the corresponding uv such that, if z is a point of xy and t is the corresponding point of uv, Dist $(u,t)=$ Dist $(x,z)/$Dist (x,y). Then to each point of S there corresponds exactly one point of Q. In general the converse is not true, for two or more different segments $\{xy\}$ may have common points.

We proceed to prove that Q is the continuous image of S. Let $t_i \to t$ in S and let z_i and z be the corresponding points in Q; we must prove that $z_i \to z$. Let t_i lie on u_iv_i, t lie on uv, x_iy_i correspond to u_iv_i, and xy correspond to uv. It is obvious that $u_i \to u$, $v_i \to v$, whence $x_i \to x$, $y_i \to y$, and $x_iy_i \to xy$. Hence, whether or not the sequence $\{z_i\}$ converges, $\overline{\lim}\, z_i \subset xy$.

By construction Dist (x_i, z_i) = Dist $(x_i, y_i) \times$ Dist (u_i, t_i). By the previous paragraph Dist $(x_i, y_i) \to$ Dist (x, y) and also Dist $(u_i, t_i) \to$ Dist (u, t). Since Q is bounded, the sequence $\{z_i\}$ has a convergent sub-sequence $\{z_i'\}$ whose limit we call z'. The results just obtained show that Dist (x, z') = Dist $(x, y) \times$ Dist (u, t), or Dist (u, t) = Dist (x, z')/Dist (x, y). The definition of the correspondence between Q and S gives immediately that $z' = z$. As this holds for every convergent sub-sequence of $\{z_i\}$, it follows that $z_i \to z$ and hence that the correspondence is continuous.

Let $A' = A \cdot Q$ and $B' = B \cdot Q$. Then $A' \cdot bd = 0$, $B' \cdot ac = 0$, and $(A' + B') \cdot (ab + cd) = 0$. Let E be the set of all the points of S whose images in Q are points of A', and let F have a like relation to B'. Then $E \cdot pr \neq 0$, $F \cdot qs \neq 0$, $E \cdot qs = 0$, $F \cdot pr = 0$ and $(E + F) \cdot (pq + rs) = 0$. Since the correspondence is continuous, E and F are closed; since $A' \cdot B' = 0$, $E \cdot F = 0$.

As $E + F$ does not contain a continuum joining pr to qs, it follows from the properties of the euclidean plane that there is a continuum K in $S - (E + F)$ such that $K \cdot rs \neq 0$ and $K \cdot pq \neq 0$. Let L be the image of K in Q. Then $L \cdot (A' + B') = 0$ and L is a continuum. As $L \subset Q$, $L \cdot (A + B) = 0$. Since $K \cdot rs \neq 0$ and since $K \cdot pq \neq 0$, $L \cdot ab \neq 0$ and $L \cdot cd \neq 0$.

But $ab \subset M$ and $cd \subset N$, whence it follows that $C = A + B$ is not an $S(M, N)$, contrary to hypothesis. Thus the theorem is proved.

YALE UNIVERSITY

ON THE TRANSFORMATION WHICH LEADS FROM THE BRIOSCHI QUINTIC TO A GENERAL PRINCIPAL QUINTIC*

BY RAYMOND GARVER

There are two well known methods of showing that any quintic equation (subject to certain restrictions which will not be gone into here) can be reduced to the important Brioschi normal form

$$w^5 - 10Zw^3 + 45Z^2w + k = 0, \tag{1}$$

with the aid of no irrationalities other than two square roots. Both methods employ a preliminary transformation to reduce the given quintic to the so-called principal form

$$Y^5 + 5aY^2 + 5bY + c = 0. \tag{2}$$

The coefficients of this transformation will involve one square root, in general, but no other irrationality. One method, that devised by Gordan † and later improved by Weber‡ and Dickson§, then sets up a transformation of the form $w = R(Y)$‖ which leads from (2) to (1). This is, of course, the direct process, and the one which we should expect to follow, as a rule. However, in the present case, the required transformation is not at all simple, it is not easy to set up, and the constant term in the transformed equation is very difficult to determine explicitly.

Under the circumstances, it is by no means out of place to consider the second method, which, starting with equation(1), devises a transformation which leads from it to a principal quintic which can be identified with (2). It is then possible, by a known process, to set up a transformation leading from

* Presented to the Society, June 20, 1929.

† Mathematische Annalen, vol. 28 (1887), pp. 152–166.

‡ *Algebra*, 2d ed., vol. I, 1898, pp. 263–267.

§ *Modern Algebraic Theories*, 1926, pp. 214–218.

‖ $R(Y)$ is a rational function of Y, the coefficients of which are rational functions of the coefficients of (2) and the square root of the discriminant of (2).

(2) back to (1), provided (2) does not have a double root. This second method has several distinct advantages. It employs a considerably simpler transformation than does the first method, namely

$$Y = \frac{\lambda + \mu w}{Z^{-1}w^2 - 3}, \tag{3}$$

λ and μ being parameters and k in (1) being put equal to $-Z^2$. The coefficients of the transformed equation (2) are then obtained from

$$\begin{aligned} Va &= 8\lambda^3 + \lambda^2\mu + 72\lambda\mu^2 Z + \mu^3 Z, \\ Vb &= -\lambda^4 + 18\lambda^2\mu^2 Z + \lambda\mu^3 Z + 27\mu^4 Z^2, \\ Vc &= \lambda^5 - 10\lambda^3\mu^2 Z + 45\lambda\mu^4 Z^2 + \mu^5 Z^2, \end{aligned} \tag{4}$$

where V is written in place of $1728 - 1/Z$. Dickson shows how to identify the transformed equation with an arbitrary principal quintic by solving equations (4) for λ, μ, Z in terms of a, b, c.* The solution involves only one irrationality, again the square root of the discriminant of (2).

What this does is to tell us which particular equation (1) will transform into an arbitrary principal quintic, so we may say that the explicit form of both the original and the transformed equation is easy to determine. This is another advantage over the first method. Finally, while it is possible, as has been mentioned above, to set up, using (3), a transformation interchanging the role of our original and transformed equation, this need not be done explicitly. For it is possible that (3), which gives a root of any principal quintic in terms of a root of a certain Brioschi quintic, is a more useful relation than that given by the inverted transformation.

Until very recently, however, no direct, simple proof had been given showing that (3) led to a transformed equation with coefficients determined by (4). Gordan, who seems to be the first to obtain (3), or rather an equation essentially equivalent to it,† secured it merely as a corollary to a long discussion of other matters, chiefly icosahedral forms. A similar statement applies to Dickson's presentation. Kiepert‡ uses

* Ibid., pp. 244–247. This also covers the next reference to Dickson, below.

† Mathematische Annalen, vol. 13 (1878), pp. 375–404.

‡ Journal für Mathematik, vol. 87 (1879), pp. 114–133.

(3) and (4) in his work on the solution of the quintic by elliptic functions, but does not show how the transformed equation is set up. Heymann, in 1894, derived the transformed equation directly from the transformation, but his method is long and, in part, rather complicated.* A much improved treatment which may be based, to a certain extent, on that of Heymann was given by Perron, in 1927.† It seems to be the first simple, independent treatment of the matter.

Perron considers (1) with $Z=1$, but leaves k to be determined later. His transformation is

$$Y = \frac{\alpha}{w - 3^{1/2}} + \frac{\beta}{w + 3^{1/2}}, \tag{5}$$

which is essentially equivalent to (3). He sets up the transformed equation with the aid of values which he is able to obtain for expressions of the form

$$\sum \frac{1}{(w - 3^{1/2})^k (w + 3^{1/2})^l},$$

the summation being over the five roots of (1). Since his parameters α and β are not the λ, μ of (3), he does not obtain equations (4), but three somewhat similar equations.

The main purpose of this article is to give another treatment of the matter, one which is simple and direct and which leads to equations (4). These are, I believe, preferable to those of Perron, in that they can be solved somewhat more nicely for λ, μ, Z. We shall use (1) with $k = -Z^2$, and apply to this the transformation

$$v = w^2 Z^{-1} - 3. \tag{6}$$

The equation in v can be set up very easily by a device used in elementary work on equations; transpose the constant term of (1) to the right, factor w out of the left side, replace even powers of w, using

$$w^2 = Z(v + 3), \tag{7}$$

square both sides, replace w^2 and simplify. The result is

* Zeitschrift für Mathematik und Physik, vol. 39 (1894), pp. 162–182.

† *Algebra*, vol. 2, 1927, pp. 209–216.

$$Z(v+3)(v^2-4v+24)^2=1, \tag{8}$$

or

$$v^5-5v^4+40v^3+V=0,\ V=1728-1/Z. \tag{9}$$

The step just before (8) gives w rationally in terms of v,

$$w(v^2-4v+24)=1, \tag{10}$$

or using (8),

$$w=Z(v^3-v^2+12v+72). \tag{11}$$

We must now apply to (9) the transformation

$$Y=\frac{\lambda+\mu w}{v}, \tag{12}$$

where w must be replaced by its value in terms of v. We do not substitute (11) explicitly into (12), since (7) will be used as well. In setting up the transformed equation the sums of certain powers of the roots of (9) are needed; in the usual notation these are

$$s_2=-55,\ s_1=5,\ s_{-1}=s_{-2}=0,\ s_{-3}=\frac{-120}{V},\ s_{-4}=\frac{20}{V}. \tag{13}$$

If we now sum (12), using (11), for the roots of (9), we have, dropping at once the terms in s_{-1} and s_{-2}, $\sum Y=\mu Z(s_2-s_1+60)$, which is obviously zero. If we square (12), substitute from (7) and (11), and sum we find $\sum Y^2=2\lambda\mu Z(s_1-5)=0$. Hence the transformed equation has no terms in Y^4 and Y^3.

When (12) is cubed, w and w^2 replaced as before and w^3 by its value $Z^2(v^4+2v^3+9v^2+108v+216)$ from (7) and (11), it becomes

$$\begin{aligned} Y^3=\ &\lambda^3v^{-3}+3\lambda^2\mu Z(1+72v^{-2})+3\lambda\mu^2Z(3v^{-3}) \\ &+\mu^3Z^2(v+2+216v^{-3}) \\ &+\text{terms in } v^{-1} \text{ and } v^{-2}. \end{aligned} \tag{14}$$

Now $\sum Y^3$ can be found very easily by using (13) and reducing the terms which do not involve s_{-3} to the denominator V, using, however, in the numerator the value $1728-1/Z$ in place of V. From the relation $\sum Y^3=-15a$ we are then led to the first equation (4).

Similarly, when (12) is raised to the fourth power, and w^4 replaced by its value from (7), we have

$$(15)\quad Y^4 = \lambda^4 v^{-4} + 4\lambda^3\mu Z(12v^{-3} + 72v^{-4}) + 6\lambda^2\mu^2 Z(v^{-3} + 3v^{-4}) + 4\lambda\mu^3 Z^2(1 + 108v^{-3} + 216v^{-4}) + \mu^4 Z^2(6v^{-3} + 9v^{-4}) + \text{terms in } v^{-1} \text{ and } v^{-2}.$$

When this is summed and (13) used, every term but $20\lambda\mu^3 Z^2$ has the denominator V. Change it as above, employ the relation $\sum Y^4 = -20b$, and the second equation of (4) is obtained.

It is of course possible to determine c in the same way from $\sum Y^5$, but a simpler method is available. For c is also the negative of the product of the roots of (2), or from (12),

$$(16)\qquad c = -\frac{\prod(\lambda + \mu w)}{\prod v}.$$

Now by (9) the denominator of (16) has the value $-V$. The numerator is, by definition, the resultant of the left side of (1) and $\lambda + \mu w$. By a familar theorem on resultants this is equal to the negative of the resultant of the two polynomials taken in the other order, which gives at once the value

$$(17)\quad -\mu^5\left[\left(-\frac{\lambda}{\mu}\right)^5 - 10Z\left(-\frac{\lambda}{\mu}\right)^3 + 45Z^2\left(-\frac{\lambda}{\mu}\right) - Z^2\right].$$

The third equation of (4) follows at once from (16) and (17), and the transformed equation is completely determined.

This determination, together with Dickson's treatment of equations (4), gives a direct, self-contained proof of the theorem mentioned at the beginning of the paper. It remains simply to mention briefly the exceptional cases that may arise. For (3) to be valid, Z must not be zero, and $Z^{-1}w^2 - 3$ must not vanish for a root of (1). Both of these cases are covered if we require that (1) have no multiple root. A similar restriction will be made on (2), to insure the existence of a transformation leading from (2) back to (1). Finally, certain minor restrictions must be made in solving equations (4) for λ, μ, Z. Perron, in treating his system of equations similar to (4), mentions that the necessary restrictions characterize certain simple solvable quintics. A similar statement may be made for equations (4).

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

AN APPLICATION OF THE METHOD OF PARAMETERS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS*

BY F. H. MILLER

1. *Introduction.* The method of variation of parameters provides, as is well known, an elegant means of finding the complete primitive of a linear ordinary differential equation whose complementary function is given. In this paper it is shown that the method is applicable also to certain linear partial differential equations of the second order and that the results so obtained are interconnected with the familiar Laplace transformation.

In what follows we shall understand that the functions of x and y considered are of class C'' in a domain of definition K, where C'' is the class of functions which are continuous together with their first and second derivatives. We shall suppose that the complementary function of our differential equation is of the form $F(\alpha)+G(\beta)$, α and β being known distinct expressions in x and y, and F and G arbitrary functions, in C''. It may be readily verified that any differential equation having such a complementary function will be of the type

$$(1) \qquad (RD^2 + SDD' + TD'^2 + PD + QD')z = V,$$

where $R, S, \cdots, V$ are functions of x and y alone and $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Not every equation conforming to (1), however, will have $F(\alpha)+G(\beta)$ as complementary function; a criterion that may be easily applied arises from the following conditions,† necessary and sufficient in class C'':

$$(2a) \qquad R\alpha_x^2 + S\alpha_x\alpha_y + T\alpha_y^2 = 0,$$

$$(2b) \qquad R\beta_x^2 + S\beta_x\beta_y + T\beta_y^2 = 0,$$

$$(2c) \qquad R\alpha_{xx} + S\alpha_{xy} + T\alpha_{yy} + P\alpha_x + Q\alpha_y = 0,$$

$$(2d) \qquad R\beta_{xx} + S\beta_{xy} + T\beta_{yy} + P\beta_x + Q\beta_y = 0.$$

* Presented to the Society, October 26, 1929.

† It should be noticed that equations (2a)–(2d) place symmetrical hypotheses on α and β; this symmetry will be made use of later in the discussion.

These relations must be satisfied by α and β identically in x and y; for, if we substitute $z=F(\alpha)+G(\beta)$ in the homogeneous equation obtained from (1) by replacing V by zero, it follows that since F and G are arbitrary, the coefficients of their first and second derivatives must vanish identically in K, whence we have the above relations.

We further restrict the domain K by assuming, throughout our work, $S^2-4RT\neq 0$, $J=\alpha_x\beta_y-\alpha_y\beta_x\neq 0$, in K. In § 2 we confine our attention to the case in which the first partial derivatives of α and β are all different from zero in K, and in §3 we consider the special cases in which one or more of these derivatives vanish identically in K; it will also be found necessary in each of these two discussions to assume K restricted so that solutions of certain associated ordinary differential equations needed in the process be obtainable.*

In the last section we shall indicate the relation between our previously derived results and the process of effecting a Laplace transformation.

2. *The General Case.* The criterion mentioned in the introduction for the determination of those equations having complementary functions $F(\alpha)+G(\beta)$, when the first partial derivatives of α and β are different from zero in K, is indicated by the form of equations (2a)-(2d). We obtain the roots of the quadratic $R\theta^2+S\theta+T=0$, whose discriminant is by hypothesis different from zero in K, and denote them by α_x/α_y and β_x/β_y in accordance with equations (2a) and (2b);† the functions α and β so determined must then, by (2c) and (2d), themselves satisfy the homogeneous equation corresponding to (1).

Supposing this test fulfilled, and that we are in possession of the complementary function, we assume the general solution of (1) to be

$$z = \alpha u + \beta v + w, \tag{3}$$

where u, v, w are unknown functions of x and y in C'' to be employed as parameters. Consequently, taking

* The equations referred to are (11), (14), and the corresponding ones of §3.

† The quadratic in θ shows us that, apart from a possible common factor, $R=\alpha_y\beta_y$, $S=-(\alpha_x\beta_y+\alpha_y\beta_x)$, $T=\alpha_x\beta_x$; thus R and T are in the present case certainly different from zero in K.

$$\alpha u_x + \beta v_x + w_x = 0, \tag{4}$$

$$\alpha u_y + \beta v_y + w_y = 0, \tag{5}$$

as the two additional conditions on our parameters, we find

$$\left\{\begin{aligned} Dz &= \alpha_x u + \beta_x v, \quad D'z = \alpha_y u + \beta_y v, \\ D^2z &= \alpha_x u_x + \alpha_{xx} u + \beta_x v_x + \beta_{xx} v, \\ DD'z &= \alpha_y u_x + \alpha_{xy} u + \beta_y v_x + \beta_{xy} v, \\ D'Dz &= \alpha_x u_y + \alpha_{xy} u + \beta_x v_y + \beta_{xy} v, \\ D'^2z &= \alpha_y u_y + \alpha_{yy} u + \beta_y v_y + \beta_{yy} v. \end{aligned}\right. \tag{6}$$

Since $DD'z$ must be identical with $D'Dz$,* it follows that

$$\alpha_y u_x - \alpha_x u_y = -\beta_y v_x + \beta_x v_y. \tag{7}$$

From (4) and (5) we get $\alpha du + \beta dv + dw = 0$; combining this with dz found from (3), we obtain

$$dz = u\,d\alpha + v\,d\beta = (u\alpha_x + v\beta_x)dx + (u\alpha_y + v\beta_y)dy. \tag{8}$$

By virtue of relation (7), the last member of (8) is seen to be a total differential. Hence it is necessary merely to determine u and v, insert their values in (8), and integrate.

Substituting from (6) in the given equation (1) yields, after reduction by means of equations (2d) and (2c),

$$(R\alpha_x + S\alpha_y)u_x + T\alpha_y u_y + (R\beta_x + S\beta_y)v_x + T\beta_y v_y = V. \tag{9}$$

Now, from (2b), $-(R\beta_x + S\beta_y)/\beta_y = T\beta_y/\beta_x$; making use of this after multiplying equation (7) throughout by $-(R\beta_x + S\beta_y)/\beta_y$, we have

$$\begin{aligned} -\alpha_y(R\beta_x + S\beta_y)u_x + \alpha_x(R\beta_x + S\beta_y)u_y \\ = \beta_y(R\beta_x + S\beta_y)v_x + T\beta_y^2 v_y. \end{aligned}$$

Eliminating v_x and v_y between this equation and (9), we obtain a linear partial differential equation of first order in u, the subsidiary equations of which are

$$dx/(RJ) = dy/(R\alpha_x\beta_x + S\alpha_x\beta_y + T\alpha_y\beta_y) = du/(\beta_y V). \tag{10}$$

As an immediate consequence of (2a), we get from (10), $\alpha = c_1$,

* For z as given by (3) is by hypothesis in C''.

a constant. Solving $\alpha = c_1$ for y* in terms of x and c_1, and inserting this value of y in (10), there is obtained

$$\frac{du}{dx} = \frac{\beta_y V}{RJ}. \tag{11}$$

Integrating (11) and replacing c_1 by α in the result gives us $u = \phi(x, y) + \text{const.}$ Consequently we have

$$u = \phi(x, y) + F_1(\alpha), \tag{12}$$

where F_1 is an arbitrary function in C''.

The function v is found in precisely the same manner,† the equation determining it being

$$-RJv_x + (R\alpha_x\beta_x + S\alpha_y\beta_x + T\alpha_y\beta_y)v_y = \alpha_y V. \tag{13}$$

It follows in this case that the equations subsidiary to (13) give us, by (2b), $\beta = c_2$, and therefore

$$\frac{dv}{dx} = -\frac{\alpha_y v}{RJ}, \tag{14}$$

where again y is supposed replaced by its value from $\beta = c_2$. Hence there is found

$$v = \psi(x, y) + G_1(\beta). \tag{15}$$

Substituting from (12) and (15) in (8), we get finally

$$dz = (\phi\alpha_x + \psi\beta_x)dx + (\phi\alpha_y + \psi\beta_y)dy + F_1(\alpha)d\alpha + G_1(\beta)d\beta,$$

and

$$z = \int [(\phi\alpha_x + \psi\beta_x)dx + (\phi\alpha_y + \psi\beta_y)dy] + F(\alpha) + G(\beta), \tag{16}$$

the first member on the right representing the desired particular integral.

3. *Special Cases.* The existence of what we shall call special or exceptional cases, in which one or more first derivatives of α and β vanish identically in K, is easily recognized from the form of equation (1). For equations (2a) and (2b) show that

* This is possible since $\alpha_y \neq 0$ in K.

† It is here that the symmetrical hypotheses on α and β play a part. The function u having been found, v might also be obtained from (7); however, because of symmetry, and for use in §4, equation (13) is preferable.

the vanishing of a derivative of α or β entails the vanishing of R or T, so that if one of these derivatives is identically zero in K the term in D^2z or D'^2z will be absent from the differential equation. In such event also, the arguments α and β are found without difficulty,* and we may proceed to the evaluation of the particular integral.

(a) First suppose that instead of having $\alpha_x \neq 0$ we now have $\alpha_x \equiv 0$ in K, while all other restrictions previously imposed on K and the differential equation (1) are retained. Then, by (2a), T must be identically zero in K. Consequently equations (7) and (9) reduce to

$$\alpha_y u_x = -\beta_y v_x + \beta_x v_y, \tag{7a}$$

$$S\alpha_y u_x + (R\beta_x + S\beta_y)v_x = V, \tag{9a}$$

respectively. We may evidently combine these so as to obtain a partial differential equation of first order determining v, and knowing this function, we may find u from (7a). The case in which $\beta_x \equiv 0$ instead of $\alpha_x \equiv 0$ is precisely the same except that α and β, and u and v, are interchanged in (7a) and (9a).

(b) If, other conditions being the same, we next suppose that α_y in place of α_x is identically zero in K, equation (2a) shows us that R must vanish, and therefore equations (7) and (9) become

$$-\alpha_x u_y = -\beta_y v_x + \beta_x v_y, \tag{7b}$$

$$S\beta_y v_x + T\beta_y v_y = V. \tag{9b}$$

Equation (9b) determines v directly and (7b) then gives us u. Again, the case in which $\beta_y \equiv 0$ in place of $\alpha_y \equiv 0$ is entirely similar.

(c) Finally, suppose the conditions altered by having α_x and β_y both identically zero in K. Then

$$\alpha_y u_x = \beta_x v_y, \tag{7c}$$

$$S\alpha_y u_x = V. \tag{9c}$$

The functions u and v are readily obtainable from these relations. If α_y and β_x vanish identically in K, a like pair of equations serves to fix u and v.

* See, for example, Forsyth's *Theory of Differential Equations*, 1906, vol. vi, p. 32, cases iii–v.

Obviously both first derivatives of α (or of β) cannot vanish identically in K simultaneously, nor can we have $\alpha_x=\beta_x=0$ (or $\alpha_y=\beta_y=0$) identically in K, since by hypothesis the jacobian J does not vanish in K. Thus, the exceptional cases are all amenable to the method described in the preceding section.

4. *The Laplace Transformation.* If we take α and β as new independent variables, equation (1) reduces to

$$\partial^2 z/\partial\alpha\,\partial\beta = -\alpha_y\beta_y V/(RJ^2), \tag{17}$$

assuming the conditions of §2 fulfilled. The right-hand member of (17) may also be expressed as $-\alpha_x\beta_x V/(TJ^2)$; cases (a) and (b) of the preceding section are readily shown to submit to one or the other of these two modes of expression, while (c) obviates the necessity of any transformation, since (1) must then be simply $SDD'z=V$ by (2a)–(2d). Now if $\alpha=\alpha(x, y)$ and $\beta=\beta(x, y)$ can actually be solved* for x and y in terms of α and β, substitution of these values in the right member of (17) followed by two simple integrations yields the general solution. We shall call such a procedure a "direct" application of Laplace's transformation. In some cases, however, it is not possible to effect this substitution, as the following example illustrates.

Let there be given the differential equation

$$[x(1-x)D^2 - y(1-x^2)DD' + y^2(1-x)D'^2 + D - xyD']z = 1,$$

and suppose the domain K to consist of any region not including those points for which $x=0$ or 1 or $y=0$. Using the relations (2a)–(2d), we find that $F(xy)+G(ye^z)$ is the complementary function. Now it is a practical impossibility to solve the equations $\alpha=xy$, $\beta=ye^z$ simultaneously for x and y, and consequently Laplace's transformation cannot be directly applied. But equations (11) and (14) are here $du/dx=1/[c_1(1-x)^2]$, $dv/dx=-1/[c_2(1-x)^2]$, so that $\phi=1/[xy(1-x)]$, $\psi=-1/[ye^z(1-x)]$. Substitution in (16) therefore gives as the general solution $z=\log x+F(xy)+G(ye^z)$.

Thus, when a given differential equation of the type here considered does not lend itself to a direct application of (17), we may nevertheless obtain the general solution from equation

* Of course, this can theoretically always be accomplished, aside from algebraic difficulties, under our hypotheses.

(16). I am indebted to H. W. Reddick for pointing out to me that (16) may also be viewed as a consequence of equation (17); that is, the result obtained from the standpoint of the parametric method is also intimately connected with the Laplace transformation. Since

$$\alpha_x = Jy_\beta,\ \alpha_y = -Jx_\beta,\ \beta_x = -Jy_\alpha,\ \beta_y = Jx_\alpha, \tag{18}$$

equation (17) gives us*

$$\begin{aligned} \frac{\partial z}{\partial \alpha} &= \int_\beta \frac{\beta_y V}{RJ}\frac{\partial x}{\partial \beta}\,d\beta + F_1(\alpha), \\ \frac{\partial z}{\partial \beta} &= -\int_\alpha \frac{\alpha_y V}{RJ}\frac{\partial x}{\partial \alpha}\,d\alpha + G_1(\beta), \end{aligned} \tag{19}$$

where the subscript on an integral sign denotes the variable with respect to which the integration is to be performed. But when we integrate with respect to β, $x_\beta d\beta$ is merely dx; likewise, in the second of equations (19), $x_\alpha d\alpha$ will be dx. Hence the above two integrals are simply the expressions denoted by ϕ and ψ respectively, and therefore

$$\begin{aligned} z &= \int (z_\alpha d\alpha + z_\beta d\beta) + F(\alpha) + G(\beta) \\ &= \int (\phi d\alpha + \psi d\beta) + F(\alpha) + G(\beta), \end{aligned}$$

which is our equation (16).

Theoretically, of course, the solution of equation (1) is expressed by writing

$$z = \int_\beta \int_\alpha \frac{V}{R}\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta}\,d\alpha d\beta + F(\alpha) + G(\beta), \tag{20}$$

which follows directly from (17); our purpose here has been to indicate the manner in which the particular integral of (20) may be explicitly determined. Although, as we have seen, our method of attack may be viewed in connection with a Laplace transformation, it is interesting to observe that these results also follow from the familiar idea of variation of parameters.

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* We are again supposing the hypotheses of §2 to be satisfied. In the exceptional cases (17) may be taken in another form, as previously stated; the details need not be considered anew.

NOTE ON SEPARABILITY*

BY R. G. PUTNAM

The following theorems have been shown by R. L. Moore† to hold in a class D of Fréchet.‡

THEOREM 1. *In order that every subclass of a given class D of Fréchet should be separable, it is necessary and sufficient that every uncountable subclass of that class D should have a limit point.*

THEOREM 2. *If D_0 is a separable class D, then every uncountable subclass of D_0 contains a point of condensation.*

THEOREM 3. *Every subclass of a separable class D is itself separable.*

THEOREM 4. *In order that every uncountable subclass of a given class D should contain a point of condensation of itself, it is necessary and sufficient that every uncountable subclass of D should have a limit point.*

THEOREM 5. *In order that every ascending sequence of distinct closed subsets of a given class D should be countable, it is necessary and sufficient that every descending one should be.*

Theorems 3 and 4 follow from Theorems 1 and 2, and 5 is obtained with the aid of Theorems 1 and 4.

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* Presented to the Society, September 6, 1928.

† Fundamenta Mathematicae, vol. 8, p. 189. Theorems 1, 2, and 3 have been previously considered by W. Gross in *Zur Theorie der Mengen, in denen ein Distanzbegriff definiert ist*, Sitzungsberichte, Wien, vol. 123 (1914), pp. 801–819. See also a reference to this article in *An acknowledgement*, by R. L. Moore, Fundamenta Mathematicae, vol. 8, p. 374.

‡ A class D of Fréchet is a class of elements which satisfy the following conditions:

1. With every pair of elements A and B there is associated a number $(A, B)=(B, A)\geqq 0$.

2. $(A, B)=0$ if, and only if, $A=B$.

3. If A, B and C are any three elements, then $(A, C)\leqq(A, B)+(B, C)$.

4. The sequence of elements $P_1, P_2, P_3, \cdots$ converges to a limit P if and only if the distance (P, P_n) approaches zero as n becomes infinite. A class in which conditions 1, 2 and 4 hold but in which 3 need not hold is a class E.

The following example, due to P. Alexandroff, shows that none of the first four theorems stated above is true without change in all classes E. Consider the space composed of two parallel lines A and B. Let the écart of any two distinct points x and y both of which are on A or on B be unity while the écart of a point x on A and a point y on B is the ordinary distance from x to the projection of y on A or vice versa; if x and y are not distinct their écart is zero. In this space the limit points of a set entirely on one line are all on the other line. The space thus formed is a separable class E. Every uncountable subclass has a limit point but is not always separable and does not contain a point of condensation. Theorem 4 is not true in this space.

Theorems 1′, 2′, 4′, and 5′ which follow, are modifications respectively of Theorems 1, 2, 4, and 5, and can be shown to hold in classes E (not at the same time classes D).

THEOREM 1′. *In order that every subclass of a given class E should be separable, it is necessary and sufficient that every uncountable subclass of that class E should contain a limit point of itself.*

The proof of this theorem is the same as the proof of Theorem 1 in Professor Moore's paper.

THEOREM 2′. *If every subclass of a given class E is separable, then every uncountable subclass of that class E contains a point of condensation of itself.*

The proof of this theorem follows from the first part of the proof of Theorem 2.

Theorem 3 does not hold in all classes E as the above example shows.

THEOREM 4′. *In order that every uncountable subclass of a given class E should contain a point of condensation of itself, it is necessary and sufficient that every uncountable subclass of the given class should contain a limit point of itself.*

This theorem follows at once from Theorems 1′ and 2′.

THEOREM 5′. *In order that every ascending sequence of distinct closed subsets of a given class E in which every derived set is closed should be countable, it is necessary and sufficient that every descending one should be.*

This theorem is a consequence of Theorems 1′ and 4′ and the result of Sierpinski, used by Professor Moore in the proof of Theorem 5.

NEW YORK UNIVERSITY

NOTE ON A SCHOLIUM OF BAYES

BY F. H. MURRAY

In his fundamental paper on a posteriori probability,* Bayes considered a certain event M having an unknown probability p of its occurring in a single trial. In deriving his a posteriori formula he assumed that all values of p are equally likely, and he recommended this assumption for similar problems in which nothing is known concerning p. In the corollary to proposition 8 he derives the value

$$\int_0^1 \binom{n}{x} p^x(1-p)^{n-x}dp = \frac{1}{n+1}$$

for the probability of x successes in n trials. This result is independent of x; in a scholium he observes that this consequence is what is to be expected, on common sense grounds, from complete ignorance concerning p, and this concordance is considered to justify the assumption that all values of p are equally likely.†

In order to complete the argument of the scholium it is necessary to show that no other frequency distribution for p has the same property.

More precisely, given that a cumulative frequency function $f(p)$ has the property that for $0 \leqq x \leqq n$, x, n being integers,

$$\int_0^1 \binom{n}{x} p^x(1-p)^{n-x}df(p) = \frac{1}{n+1},$$

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* Bayes, *An essay towards solving a problem in the doctrine of chances*, Philosophical Transactions of the Royal Society, vol. 53 (1763), pp. 370–418.

† In other words, the assumption "all values of p are equally likely" is *equivalent* to the assumption "any number x of successes in n trials is just as likely as any other number y, $x \leqq n$, $y \leqq n$." It has been suggested verbally by Mr. E. C. Molina that this proposition has a possible importance in certain statistical questions.

it is required to determine $f(p)$ from this equation. Now if $n=x$, the equation becomes

$$\int_0^1 p^x df(p) = \frac{1}{x+1}$$

consequently the moments of $f(p)$ are known. The function $f(p)$ can be completely calculated from these moments with the aid of a theorem of Stieltjes.*

Let

$$F(z) = \int_0^1 \frac{df(p)}{p+z} = \frac{1}{z}\int_0^1 \frac{df(p)}{1+\frac{p}{z}} \qquad (|z|>2)$$

$$= \frac{1}{z}\left[\int_0^1 df - \frac{1}{z}\int_0^1 p\,df + \frac{1}{z^2}\int_0^1 p^2 df - \frac{1}{z^3}\int_0^1 p^3 df + \cdots\right].$$

If f is the function already discussed, this becomes

$$F(z) = \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \frac{1}{4z^4} + \cdots$$

$$= \log\left(\frac{z+1}{z}\right).$$

Consequently the function f satisfies the equation (for $|z|>2$)

$$\log\left(\frac{z+1}{z}\right) = \int_0^1 \frac{df(p)}{p+z}.$$

From the theorem of Stieltjes, if $\psi(x)$ is a non-decreasing function of x, and

$$F(z) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{z+x},$$

then

$$\frac{\psi(\xi-0)+\psi(\xi+0)}{2} - \frac{\psi(a-0)+\psi(a+0)}{2}$$

$$= \lim_{\eta=+0} R\left(\frac{1}{\pi i}\int_{-\xi-i\eta}^{-a-i\eta} F(z)dz\right).$$

* Stieltjes, *Récherches sur les fractions continues*, Annales de Toulouse, vol. 8 (1894), pp. 172–175. Also, Perron, *Die Lehre von den Kettenbrüchen*, p. 372.

Now the function $F(z)=\log\{(z+1)/z\}$ can be defined on the real axis by continuation, hence the limits above and below the real axis are uniquely determined. Suppose ξ, a on the segment $0<a<\xi<1$.

Then

$$\int_{-\xi-i\eta}^{-a-i\eta}[\log(z+1)-\log z]dz$$

$$=\int_{-\xi}^{-a}[\log(1+x-i\eta)-\log(x-i\eta)]dx$$

$$=\Big[\begin{matrix}(1+x-i\eta)\log(1+x-i\eta)-(1+x-i\eta)\\ -(x-i\eta)\log(x-i\eta)+(x-i\eta)\end{matrix}\Big]_{-\xi}^{-a}.$$

Now

$$(1+x-i\eta)\log(1+x-i\eta)-(1+x-i\eta)$$

approaches real limits, for $x=-a$, $x=-\xi$, as $\eta\to 0$, hence makes no contribution to the sum required. We have only to consider

$$-(-a-i\eta)\log(-a-i\eta)+(-\xi-i\eta)\log(-\xi-i\eta).$$

Now as $\eta\to 0$, $-\xi-i\eta\to-\xi$. Since the approach is from below the axis of reals, and since the argument of $\log z$, like that of $\log(1+z)$, is zero for a real positive z, the argument here is $-i\pi$. Hence this sum becomes

$$(a+i\eta)[-\pi i+\log(a+i\eta)]-(\xi+i\eta)[-\pi i+\log(\xi+i\eta)].$$

This approaches the limit, as $\eta\to 0$,

$$\pi i(\xi-a)+a\log a-\xi\log\xi.$$

Hence

$$\lim_{\eta\to 0}R\left[\frac{1}{\pi i}\int_{-\xi-i\eta}^{-a-i\eta}F(z)dz\right]=\xi-a.$$

Substituting in the identity, we find

$$\frac{\psi(\xi-0)+\psi(\xi+0)}{2}-\frac{\psi(a-0)+\psi(a+0)}{2}=\xi-a,$$

or

$$\frac{\psi(\xi - 0) + \psi(\xi + 0)}{2} = \xi + \text{const.}$$

Consequently ψ itself is continuous, $0 < \xi < 1$.

Now if $a > 1$, $\xi > 1$, the integral

$$\int_{-\xi}^{-a} [\log(z + 1) - \log z] dz$$

is seen to be real, hence

$$\tfrac{1}{2}[\psi(\xi - 0) + \psi(\xi + 0)] - \tfrac{1}{2}[\psi(a - 0) + \psi(a + 0)] = 0.$$

The same is true if both a and ξ are negative.

There are three additive constants yet to be determined, one on each of the intervals $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$. If it is assumed that $\psi(-\infty) = 0$, $\psi(+\infty) = 1$, and ψ is a non-decreasing function,

$$\begin{aligned}\psi(+\infty) - \psi(-\infty) = 1 = {} & \psi(+0) - \psi(-0) \\ & + \psi(1 - 0) - \psi(+0) \\ & + \psi(1 + 0) - \psi(1 - 0).\end{aligned}$$

The central term being one, the two remaining terms vanish. Hence $\psi(-0) = \psi(+0) = 0$, $\psi(1+0) = \psi(1-0) = 1$. Finally

$$\psi(\xi) = \begin{cases} 0, & \text{if} \quad \xi = 0, \\ \xi, & \text{if} \quad 0 < \xi < 1, \\ 1, & \text{if} \quad 1 < \xi. \end{cases}$$

AMERICAN TELEGRAPH AND TELEPHONE COMPANY

A CORRESPONDENCE CONNECTED WITH A PENCIL OF CURVES OF ORDER n

BY A. R. WILLIAMS

All curves of order n passing through $\frac{1}{2}n(n+3)-1$ points pass through $\frac{1}{2}(n-1)(n-2)$ other points. Or we may say that the n^2 base points of a pencil of curves of order n are determined by the number given above. When $\frac{1}{2}n(n+3)-2$ are fixed and another moves on a curve of order m, the locus of the remaining $\frac{1}{2}n(n-1)(n-2)$ is a curve of order $m(n^2-1)$ which has a multiple point of order mn at each of the fixed points. The order of the locus is reduced by n for each passage of the given curve through a fixed point.* It is the purpose of this paper to discuss the locus of the remaining base points of a pencil when a number of them are fixed and the others necessary to determine the pencil are taken consecutive on some curve.

We consider first the case when $\frac{1}{2}n(n+3)-3$ points are fixed. Thus for example taking $n=4$, we fix 11 base points of a pencil of quartics and let 2 be consecutive on a line l. To find the locus of the remaining three we use the rational quintic surface with a double cubic whose plane sections correspond to the ∞^3 quartics through the 11 fixed points. To l corresponds on the surface a rational twisted quartic C_4; and to a pencil of quartics through the 11 points and tangent to l corresponds a pencil of plane sections whose axis is tangent to C_4. This tangent meets the quintic surface in 3 more points which correspond to the three remaining base points of the pencil of plane quartics. The tangents to C_4 form a developable of order 6. Therefore, the plane curve which corresponds to the intersection of the developable with the quintic is of order 6×4 and has a sextuple point at each of the 11 fixed points. In this locus the line l is counted twice. Hence the locus sought is of order 22. In addition to the singularities at the fixed points

* Milinowski, Journal für Mathematik, vol. 67, p. 263. For $n=3$ we have the Geiser involutorial transformation of order 8.

it has a number of nodes which correspond to those intersections of the double curve of the developable with the quintic which do not take place on the cuspidal edge C_4. The developable has 4 stationary tangent planes and a double curve of order 6. Four tangents to the cuspidal edge meet it again. The points of contact of the stationary planes are ordinary points on both the double curve and the cuspidal edge. A point in which the cuspidal edge is met by a tangent to itself is a cusp on the double curve and an ordinary point on the cuspidal edge. Hence in the present instance the double curve of the developable meets the quintic in $6\times5-4-2\times4=18$ points which are nodes on the common curve of the developable and the quintic, and which give nodes on the locus we are considering. The double cubic of the quintic surface meets the developable in four points not on the cuspidal edge. These are nodes on their common curve, but do not of course give nodes on the plane locus of order 22. The genus of the latter is, therefore, $210-11\times15-18=27$. Since 4 tangents to the cuspidal edge C_4 meet it again, there are 4 positions of the consecutive pair on l such that one of the remaining 3 falls at a distinct point on l. The other 18 intersections of the locus studied with l are self-corresponding. At such a point the pencil of quartics have 3-point contact; and on the surface C_4 touches one of the principal tangents to the quintic, and the osculating plane to C_4 is the tangent plane to the quintic. The plane locus sought and its counterpart on the rational surface must always have the same genus. In this case they are both of order 22. We have seen that the singularities of the plane curve are the 11 sextuple points ($=165$ nodes) and 18 nodes corresponding to 18 nodes on the space curve. The further singularities of the latter are the 4 nodes lying on the double cubic and 161 apparent double points. The last number may be verified independently by use of the surface which is the locus of a point the intersection of whose polar planes with respect to the quintic and the developable meets a given line.

Similarly, we may fix $\frac{1}{2}n(n+3)-3$ base points of a pencil of curves of order n and let 2 more be consecutive on a curve c of order m and genus p, having d nodes and k cusps apart from the fixed points, and having a multiple point of order s_i at a fixed base point A_i. Then

$$d+k+\tfrac{1}{2}\sum s_i(s_i-1)=\tfrac{1}{2}(m-1)(m-2)-p.$$

Since only two points are taken consecutive on c, the occurrence of a cusp on the latter causes no difficulty. At the cusp we have merely a pencil of curves whose common tangent is the cuspidal tangent. To find the locus of the remaining $\frac{1}{2}(n-1)(n-2)$ points we use the rational surface of order $\frac{1}{2}n(n-3)+3$, whose plane sections correspond to the ∞^3 n-ics through the fixed points. To c corresponds a curve of order $mn-\sum s_i$ and genus p, having d nodes and k cusps. Hence we have $r=2(mn-\sum s_i-1)-k+2p$, where r is the order of the developable of its tangents. A curve in 3-space of order μ and genus p, without nodes or cusps, has $2(\mu-2)(\mu-3)+2p(\mu-6)$ tangents that meet it elsewhere. If an apparent double point is replaced by a node, this number is diminished by 4; but if an apparent double point is replaced by a cusp, the reduction is equal to the order μ.* Such tangents meet the curve again in points that are cusps on the double curve of its developable. Replacing d by its value, we have for the number, γ, of such tangents and points

$$\gamma=2(mn-\sum s_i)^2+(mn-\sum s_i)(2p-10-k)+2\sum s_i(s_i-1)\\ -2m(m-3)+4k-8p+8.$$

The genus of a plane section of the developable is the same as that of the cuspidal edge. Hence the order, x, of the double curve is $\frac{1}{2}(r-1)(r-2)-p-(mn-\sum s_i)$. And the number, α, of stationary planes or the number of stationary tangents of a plane section, is

$$3(r-2)-2(mn-\sum s_i)+6p.$$

The order of the locus sought is $rn-2m$. At a fixed point A_i it has a multiple point of order $r-2s_i$. Hence the singularities at the fixed points are equivalent to $\frac{1}{2}\sum(r-2s_i)(r-2s_i-1)$ double points. The other nodes are given by the intersections of the double curve of the developable with the rational surface that do not occur on the cuspidal edge. Their number is, therefore, $x[\frac{1}{2}n(n-3)+3]-\alpha-2\gamma-4d-2k$. The genus is thus found to be

$$\tfrac{1}{2}(n-3)(5mn^2-5n\sum s_i-6m-5n-2kn)+\tfrac{1}{2}p(5n^2-15n+2),$$

* See Severi, Istituto Lombardo Rendiconti, vol. 21 (1921), p. 251; and Salmon, *Solid Geometry*, vol. 2, §617.

which reduces to p when $n=3$. This is necessary since if 6, or indeed any number, of base points of a pencil of cubics are fixed and enough more to make 8 are taken consecutive on a curve c, the locus of the 9th will in general correspond birationally to c. In special cases, the correspondence is not birational, but to any point of c will always correspond a single point of the locus of the 9th. So the genus of the latter will never be greater than the genus of c.*

The above may immediately be extended to the general case when $\frac{1}{2}n(n+3)-r$ points are fixed. But one difficulty is encountered as shown by the following simple example. Fixing 5 base points of a pencil of cubics and letting 3 be consecutive on a conic c_2, to find the locus of the 9th we use the rational quartic surface in S_4 whose hyperplane sections correspond to the ∞^4 cubics through the fixed points. To c_2 corresponds a rational sextic C_6, and to a pencil of cubics through the fixed points and osculating the conic at P correspond a pencil of hyperplane sections having in common the osculating plane to C_6 at P'. This osculating plane meets the quartic surface in one more point which corresponds to the 9th base point of the pencil of cubics. The order of the hyperdevelopable of the osculating planes to C_6 is 12. Hence the locus sought is of order $12\times3-3\times2=30$ and has a multiple point of order 12 at each fixed point. There are 18 osculating planes to C_6 that meet it elsewhere. Hence 42 intersections of the locus of the 9th with c_2 are self-corresponding, and the others correspond to distinct points on c_2. The singularities at the fixed points are equivalent to 330 double points; and since the locus must be rational, there are 76 other nodes which correspond to those intersections of the double surface of the hyperdevelopable with the quartic that do not occur on the cuspidal edge. When $n>3$, it would be necessary to determine this number independently. The characteristics of the hyperdevelopable and its double surface, and the order of its triple curve are easily obtained by projection and section. But the double surface contains the cuspidal edge, which of course lies on the rational surface. This makes it difficult in general to find the remaining intersections of the two surfaces. Moreover, the number of

* Picard, *Traité d'Analyse*, vol. 2, Chap. XV, §19.

dimensions of the double variety increases with the number of dimensions of the space in which the configuration is placed. The question thus raised is not of primary importance for the present purpose, and therefore I will not attempt in this paper to give a general formula for the genus of the locus sought when $n>3$ and the number of fixed points is less than $\frac{1}{2}n(n+3)-3$. But in all cases it is easy to find its order, its singularities at the fixed points, and the number of positions of the consecutive set on c such that one of the remaining points of the pencil is a distinct point of c.

Thus we fix $\frac{1}{2}n(n+3)-r$ points of a pencil of n-ics and let $r-1$ be consecutive on an appropriate curve c which we suppose to have no cusps or multiple points with coincident tangents. To find the locus of the remaining $\frac{1}{2}(n-1)(n-2)$ we use the rational surface of order $\frac{1}{2}n(n-3)+r$ in r-space. To c will correspond on the surface a curve C of order μ determined as above. The hyperdevelopable composed of the S_{r-2} having $(r-1)$-point contact with C meets the rational surface again in the curve which corresponds to the locus that we seek. To find the order of this hyperdevelopable, or the number of such S_{r-2} that meet a general line in S_r, we project successively on an S_{r-1} and an S_{r-2}. It follows immediately that the required number is the number of S_{r-3} (in the projection space S_{r-2}) that have stationary, that is, $(r-1)$-point, contact with C', the projection of C on the S_{r-2}. The number of such stationary S_{r-3} is the number of stationary tangents to a plane section of the hyperdevelopable of C'. This in turn is $3(R-2)-2K+6p$, where R is the order of the plane section, K the number of its cusps, and p the genus of c, and hence of C and of all curves derived from C by projection and section. But R is the number of S_{r-4} in S_{r-2} that have $(r-3)$-point contact with C' and meet a given line. And K is the number of S_{r-5} in S_{r-2} that have $(r-4)$-point contact with C' and meet a general plane; that is, the number of S_{r-5} in S_{r-3} that have $(r-4)$-point contact with C'' and meet a general line in S_{r-3}. C'' is of course the projection of C' on the S_{r-3}. Hence if F_r^{μ} denotes the order of the developable of C in S_r, we have $F_r^{\mu}=3(F_{r-2}^{\mu}-2)-2F_{r-3}^{\mu}+6p$. We have independently $F_3^{\mu}=2(\mu-1)+2p-k$, where k is the number of cusps on C. Hence in general we have

$$F_r^\mu = (r-1)(\mu-r+2)+(r-1)(r-2)p-(r-2)k.$$

We must suppose that there are no cusps on c for $r>3$. Hence $k=0$. The above result includes the nodes of C; that is, there is no difference in this respect between a node and an "apparent" double point. The order of the locus of the $\frac{1}{2}(n-1)(n-2)$ points is therefore $nF_r^\mu-(r-1)m$, where m is the order of c. At a fixed point A_i it has a multiple point of order $F_r^\mu-(r-1)s_i$, where s_i is the order of A_i on c. To find the number of positions of the consecutive set on c such that one of the remaining points falls at a distinct point on c we must find the number of S_{r-2} in S_r that have $(r-1)$-point contact with C and meet it elsewhere. For a rational curve of order μ without multiple points in S_r (and not contained in a space of fewer dimensions) this number is easily found by the principle of correspondence to be $(r-1)(\mu-r+1)(\mu-r)$. Following the method indicated by Severi in the article referred to, I have found for a similar curve of genus p the number

$$(r-1)(\mu-r+1)(\mu-r)+[\mu(r-1)(r-2)-r(r-1)^2]p.$$

Moreover, this number is reduced by $2(r-1)$ when the curve acquires a node without changing its genus.

This method when applied in very simple cases sometimes brings out a fact that is not otherwise obvious. Thus we considered pencils of cubics through 5 fixed points and osculating a conic. If we replaced the conic by a line, the locus of the 9th point would be the conic determined by the fixed points. This method shows that as the point of osculation describes the line, the 9th point describes the conic triply. For to the line corresponds on the rational quartic surface in S_4 a cubic. The 3-space of this cubic meets the quartic surface again in a line which is the image of the conic through the fixed points and a bisecant of the cubic. Through a general point of this line pass three osculating planes to the cubic. The above statement immediately follows.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 22, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 1. An invited address will be given by Mr. E. C. Molina on *The theory of probabilities.*

STANFORD UNIVERSITY, CALIFORNIA, April 5, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than March 8.

NEW YORK CITY, April 18-19, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than March 28. The Committee on Program is arranging a symposium on Differential Geometry, in which Professors Oswald Veblen, L. P. Eisenhart, W. C. Graustein, and others will participate.

CHICAGO, April 18-19, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 28. An invited address will be given by Professor Henry Blumberg on *Methods in point sets and the theory of real functions.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, Menasha, Wis., or 501 West 116th Street, New York.

Advertising space is available in the BULLETIN at $16 per page, $9 per half page, $5 per quarter page, per issue. Address correspondence regarding advertising to H. L. RIETZ, University of Iowa, Iowa City, Iowa.

Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

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